Order and Chaos in Planetary Systems

The dynamical stability of the Solar System is one of the oldest problems studied in astronomy. The motion of more than two bodies relative to each other due to their mutual gravitational attraction cannot be described analytically. Instead, the equations of motion need to be integrated using numerical methods in order to predict the orbital evolution of the system. In this way it has been established that the orbital elements of the eight major planets in the Solar System change periodically by very small amounts. For example, the semimajor axis of the Earth ($\sim 150 \cdot 10^6$ km) was shown to vary by only a few km over millions of years, and the eccentricities of the planets stay below 0.1 on similar timescales (except for the planet Mercury, which can reach up to 0.25, Laskar et al. 1992, Icarus 95, 148). However, evidence for chaotic behaviour of the orbits of the terrestrial planets has been found when integrating for billions of years (e.g. Laskar & Gastineau 2009, Nature 459, 817). For test particles with negligible mass, chaotic orbits occur for the cases when the orbital period is in resonance with that of a giant planet (i.e. the ratio of the periods is a ratio of integer numbers), or when the orbit crosses that of a giant planet. An example for the latter case is shown in Fig. 1 for the comet Chiron (Dones et al. 1996, ASP Conference Proceedings 107, 233).

![Figure 1: The future evolution of the semimajor axis of the orbit of the Saturn- and Neptune-crossing comet Chiron. The initial orbital elements of the 11 integrations shown differed by about 1 part in $10^6$. The strong dependence of the outcome on initial conditions suggests a highly chaotic orbit.](image)

The discoveries of extrasolar planetary systems that started about 20 years ago have dramatically increased the parameter space of planetary orbital elements to investigate. Of the more than 2500 systems currently known, about 20% have more than one planet, including one system with seven planets. Most of them were detected by the Kepler space mission (http://kepler.nasa.gov/). These systems are very different from the Solar System – they tend to be compact, featuring giant planets on small orbits with periods of only a few days (i.e. their orbits would lie inside the orbit of Mercury). Also, eccentricities up to 0.8 are not uncommon. Figure 2 shows the distribution of periods and sizes for some of the systems with four or more planets detected by the Kepler mission. Period ratios between planet pairs seem to peak at certain low-order resonances, suggesting that such configurations are more stable than others.
Figure 2: Some of the systems with four or more planets detected by Kepler. The size of the planet symbols is proportional to planet radius. Within each system, the largest planet is shown in red, and the smallest planet in dark blue. The positions of Mercury (88 d) and Venus (224 d) are indicated above the x-axis.

In this project we aim at exploring the stability (regular or chaotic behaviour) of planetary orbits in dense packed planetary systems such as those shown in Fig. 2. In addition we plan to investigate the areas in orbital parameter space in the Solar System which contain chaotic orbits of small bodies such as asteroids. Various codes for numerical integration of the N-body problem have been used in the past for similar purposes. A few references are Hut et al. (1995, Astrophys. J. 443, L93), Chambers (1999, Mon. Not. R. Astron. Soc. 304, 793), Richardson et al. (2000, Icarus 143, 45), Grimm & Stadel (2014, Astrophys. J. 796, 23), and Rein & Spiegel (2015, Mon. Not. R. Astron. Soc. 446, 1424).

A novel aspect that will be investigated in this project is the development or adaptation of validated methods for integration. These methods allow one to take into account the uncertainties affecting the integration, such as uncertainties in the initial conditions (starting set of orbital parameters) or truncation errors due to the approximations employed in the numerical method.

The main idea behind this class of validated integrators is that they are based on high-order Taylor series rather than constructed to preserve some symplectic form. For all practical purposes these integrators are symplectic since the local errors can be made smaller than the underlying floating point system can represent. The Taylor series are generated automatically (and efficiently) via automatic differentiation methods that are well suited for the class of problems at hand. Set-valued numerics is used to obtain rigorous error bounds, and can be used to adaptively control the global integration errors. A few references are Moore (1966, Interval Analysis. Prentice-Hall), Jorba & Zou (2005, Exp. Math., 14:1), Berz, Makino & Hoefkens (2001, Nonlinear Anal., 47:1).
Such a method will provide information regarding uncertainties for the orbital parameters at each step of integration and allow for a more realistic assessment of the orbital evolution and possible chaotic behaviour of the system. Several alternatives will be explored with the aim to establish the method which is most suitable to study the problems given above, taking into account accuracy and computing time.

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