Inconsistency formulations for existence verification by constraint propagation

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Contents

1. Newton, inconsistency formulation.
2. Boundary rejection theory (generalizes Newton).
3. Newton again...improved

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Verification of the existence of a zero

Problem

Verify that $0 \in f(X)$.

Newton

The classical Newton existence theorem:

Theorem

Let X be compact convex set in $\mathbb{R}^n$, $\hat{x} \in X$, and $f \in \mathcal{C}^1(X, \mathbb{R}^n)$. Let all slopes $Sf(X, \hat{x})$ be regular and $\boldsymbol{J}$ a set of regular matrices with $\boldsymbol{J} \supseteq Sf(\partial X, \hat{x})$. If

$$\hat{x} - \boldsymbol{J}^{-1} f(\hat{x}) \subseteq X, \tag{1}$$

then there exists a zero of f in X.

Note: $Sf(x, y) = \int_0^1 Df\big(y + t(x - y)\big)\, dt$.

Lemma

Let X *be a convex set in* $\mathbb{R}^n$, $\hat{x} \in \mathrm{int} X$, $b \in \mathbb{R}^n$, *and* $J \in \mathbb{R}^{n,n}$ *a regular matrix, then*

$$\hat{x} + J^{-1}b \in \mathrm{int} X \iff [0, b] \cap \big(b + J(\partial X - \hat{x})\big) = \emptyset. \qquad (2)$$

Assume the conditions of the Newton theorem, and additionally the slightly stronger condition

$$\hat{x} - \boldsymbol{J}^{-1} f(\hat{x}) \subseteq \operatorname{int} X. \tag{3}$$

By the Lemma, (3) is equivalent to the disjointness condition

$$[0, f(\hat{x})] \cap \big(f(\hat{x}) + \boldsymbol{J}(\partial X - \hat{x})\big) = \emptyset. \tag{4}$$

Furthermore, if $\boldsymbol{J}$ is connected, (3) is equivalent to

$$[0, f(\hat{x})] \cap \big(f(\hat{x}) + J(\partial X - \hat{x})\big) = \emptyset \text{ for some } J \in \boldsymbol{J}, \tag{5}$$

$$\{0\} \cap \big(f(\hat{x}) + \boldsymbol{J}(\partial X - \hat{x})\big) = \emptyset. \tag{6}$$

Boundary rejection

Problem

Prove that $Y \subseteq f(X)$.

Boundary rejection

Lemma

Let Y *and* R *be sets in* $\mathbb{R}^m$*, where* Y *is connected. If*

$$Y \cap \partial R = \emptyset, \tag{7}$$

$$Y \cap R \neq \emptyset, \tag{8}$$

then $Y \subseteq \mathrm{int} R$*.*

Boundary rejection

We aim to apply previous lemma with $R = f(X)$, and therefore need to cover $\partial f(X)$.

Definition

Let X be a set in $\mathbb{R}^n$ and $f : X \to \mathbb{R}^m$. A point $x \in X$ is said to be an *extreme point* of f on X if $f(x) \in \partial f(X)$. A set $E \subseteq X$ is said to be an *extreme cover* of f on X if $f(E) \supseteq \partial f(X)$. Let $\mathcal{E}_{f,X} = \{x \in X : f(x) \in \partial f(X)\}$ and $\mathcal{EC}_{f,X} = \{E \subseteq X : f(E) \supseteq \partial f(X)\}$.

Boundary rejection

The following theorem provides sufficient conditions for a connected set Y to be contained in the image $f(X)$.

Theorem

Let X be a set in $\mathbb{R}^n$, $f : X \to \mathbb{R}^m$, $E \in \mathcal{EC}_{f,X}$, and let $Y \subseteq \mathbb{R}^m$ be connected. If

$$Y \cap f(E) = \emptyset, \tag{9}$$

$$Y \cap f(X) \neq \emptyset, \tag{10}$$

then $Y \subseteq \operatorname{int} f(X)$.

Observation

The inclusion problem $Y \subseteq f(X)$ can be reduced to an inconsistency problem by using a connected covering of $Y \cup \{f(\hat{x})\}$, where $\hat{x}$ is a non-extreme point in X.

Next, a few corollaries that follow from this observation.

Boundary rejection

Corollary

Let X *be a set in* $\mathbb{R}^n$, $f : X \to \mathbb{R}^m$, $E \in \mathcal{EC}_{f,X}$, $\hat{x} \in X \setminus E$, *and* $Y \subseteq \mathbb{R}^m$. *Let* H *be a connected set in* $\mathbb{R}^n$ *that covers* $Y \cup \{f(\hat{x})\}$. *If*

$$H \cap f(E) = \emptyset, \tag{11}$$

then $Y \subseteq \operatorname{int} f(X)$.

Boundary rejection

By letting $Y = \{0\}$ and H be a line segment between $\{0\}$ and $\{f(\hat{x})\}$, we obtain a sufficient condition for the existence of a zero:

Corollary

Let X be a set in $\mathbb{R}^n$, $f : X \to \mathbb{R}^m$, $E \in \mathcal{EC}_{f,X}$, and $\hat{x} \in X \setminus E$. If

$$[0, f(\hat{x})] \cap f(E) = \emptyset, \tag{12}$$

then $0 \in \operatorname{int} f(X)$.

Boundary rejection

If $\partial X \in \mathcal{EC}_{f,X}$ and the image $f(\partial X)$ is bounded by the image of a first order slope expansion, we obtain a corollary which is essentially equivalent to the Newton theorem:

Corollary

Let X *be a closed convex set in* $\mathbb{R}^n$, $f \in \mathcal{C}^1(X, \mathbb{R}^m)$, $\partial X \in \mathcal{EC}_{f,X}$, $\hat{x} \in \operatorname{int} X$, *and* $\boldsymbol{J} \supseteq Sf(\partial X, \hat{x})$. *If*

$$[0, f(\hat{x})] \cap \big(f(\hat{x}) + \boldsymbol{J}(\partial X - \hat{x})\big) = \emptyset, \tag{13}$$

then $0 \in \operatorname{int} f(X)$.

Boundary rejection

To use the inconsistency formulations, we need supersets of the extreme covers.

Proposition

If X *is a compact set in* $\mathbb{R}^n$, $f : X \to \mathbb{R}^n$ *is continuous on* X *and locally injective on* $\mathrm{int} X$, *then* $f(\partial X) \supseteq \partial f(X)$.

Moreover, if f is a global injection, then it preserves the boundary and the interior.

Proposition

Let X *be a compact set in* $\mathbb{R}^n$ *and* $f : X \to \mathbb{R}^n$ *a continuous injection, then* $f(\partial X) = \partial f(X)$ *and* $f(\mathrm{int} X) = \mathrm{int} f(X)$.

Boundary rejection

Next, critical points:

Lemma

Let X *be a set in* $\mathbb{R}^n$, $f : X \to \mathbb{R}^m$ *continuously differentiable on* $\mathrm{int}X$, *then*

$$\mathcal{E}_{f,\mathrm{int}X} \subseteq \mathcal{C}_{f,\mathrm{int}X}. \quad (14)$$

Thus,

Proposition

Let X *be a compact set in* $\mathbb{R}^n$, *and* $f \in C^1(X, \mathbb{R}^m)$, *then*

$$\partial X \cup \mathcal{C}_{f,\mathrm{int}X} \in \mathcal{EC}_{f,X}.$$

Boundary rejection

Theorem

Let $E \subseteq \mathbb{R}^n$ be a compact set with nonempty interior, $\hat{x} \in E$, $Y \subseteq \mathbb{R}^n$, and $f \in \mathcal{C}^1(E, \mathbb{R}^n)$. Let $H \supseteq \{Y, f(\hat{x})\}$ be connected. If

$$H \cap f(\partial E \cup C_{f,\text{int}E}) = \emptyset, \quad (15)$$

then $H \subseteq f(R_{f,\text{int}E})$.

Boundary rejection

1. Find a point $\hat{x}$ with $\mathrm{dist}(f(\hat{x}), Y) \approx 0$. Let $H \supseteq f(\hat{x}) \cup Y$ be a compact and connected set, $E = \{\hat{x}\}$.
2. Expand E and reject ∂E: Expand E until $H \cap f(\partial E) = \emptyset$, which holds true if

$$\{x \in E,\, y \in H,\, f(x) = y\} \implies x \in \mathrm{int}E,$$

   or

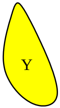
$$\{x \in \partial E,\, y \in H,\, f(x) = y\} \text{ is inconsistent.}$$

   If ∂E is rejected, we have a possibly reduced set $E' \subseteq E$.
3. Reject critical points of E': $H \cap f(C_{f,E'}) = \emptyset$ holds true if f has no critical points in E', or

$$\{x \in E',\, y \in H,\, f(x) = y,\, Df(x) \text{ is singular}\} \text{ is inconsistent.}$$

4. *If both the boundary and critical point rejections are successful, then it follows that* $H \subseteq f(R_{f,\mathrm{int}E})$.

Boundary rejection



Boundary rejection

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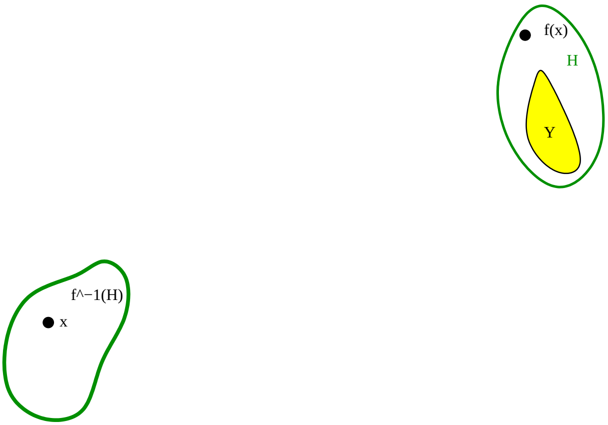
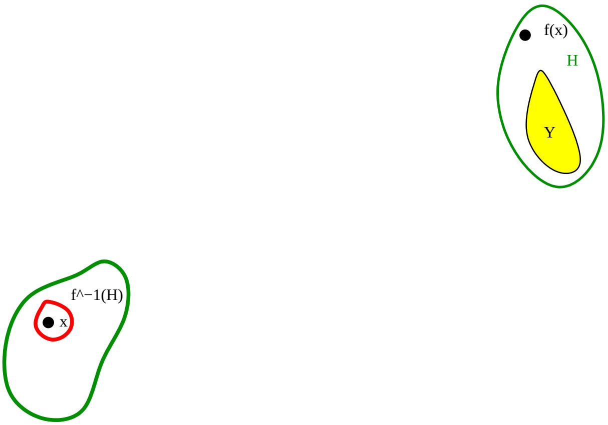


Boundary rejection



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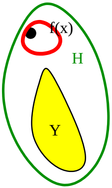

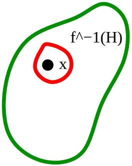

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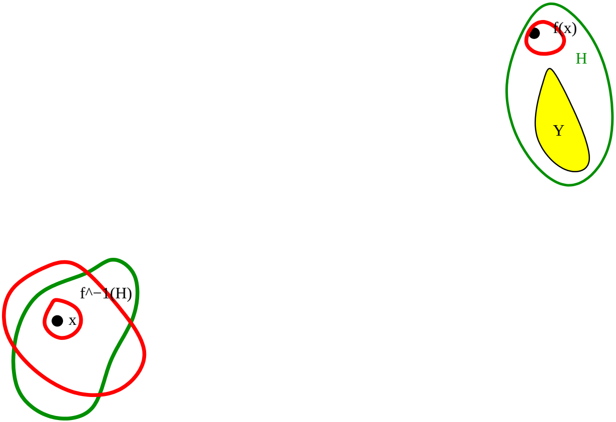

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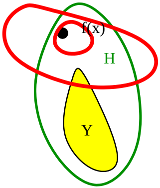
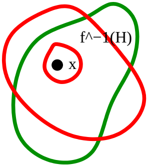
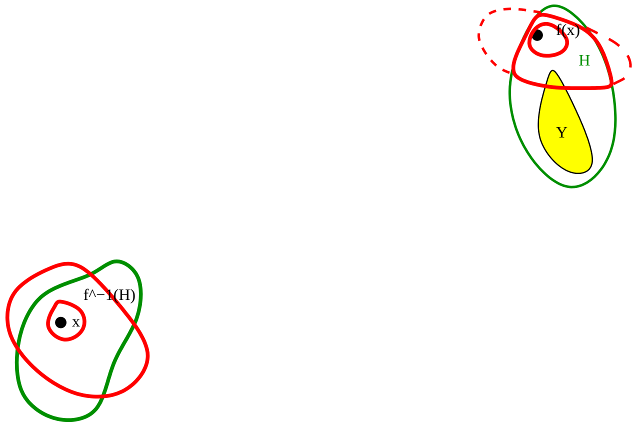
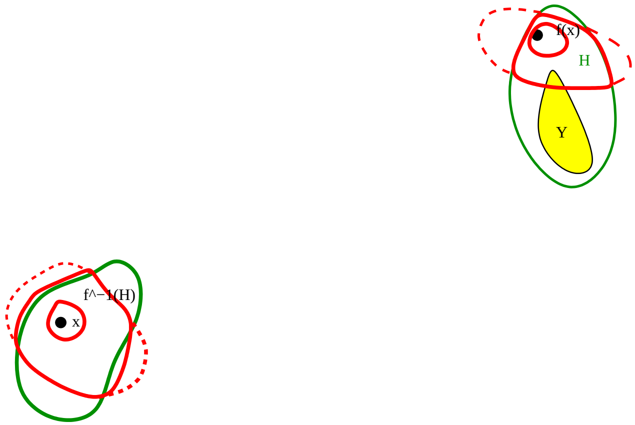




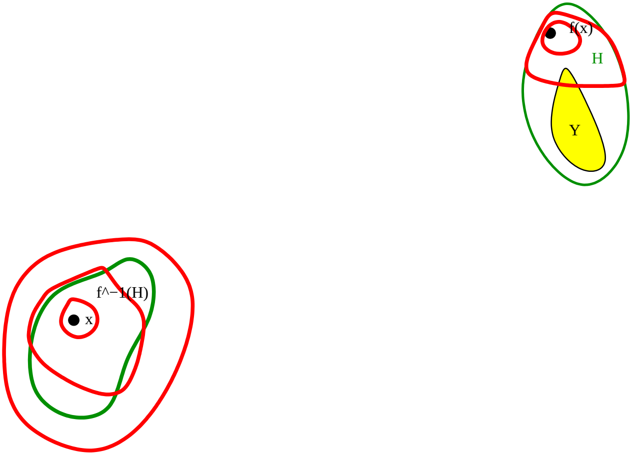
Boundary rejection



Boundary rejection

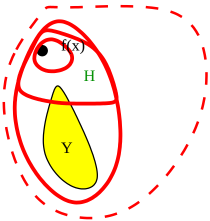


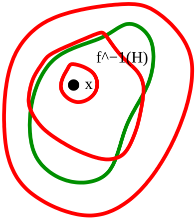
Boundary rejection



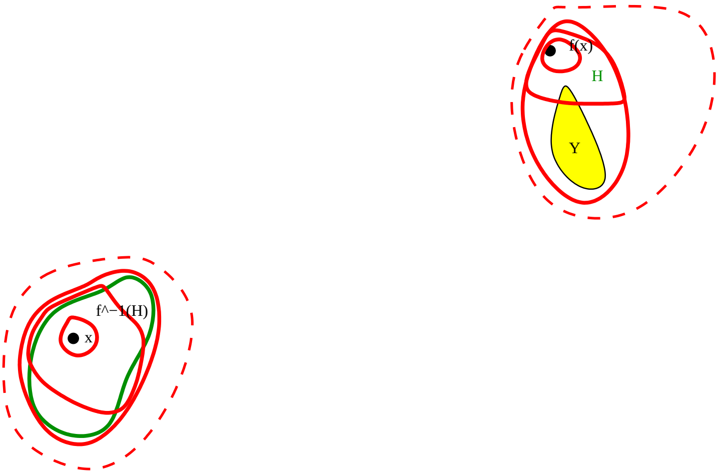
Boundary rejection

Boundary rejection





Boundary rejection



The boundary rejection formalism holds true for $\boldsymbol{J} \supseteq \mathrm{co} Df(X)$.

Does it hold true for $\boldsymbol{J} \supseteq Df(X)$?

Let's take a look at a dynamical systems approach.

Theorem

Let X a compact convex set in $\mathbb{R}^n$, and $F \in \mathcal{C}^1(X, \mathbb{R}^n)$. If

$$\hat{x} \in \mathrm{int}X,\ \hat{x} + F(X) \subseteq \mathrm{int}X, \tag{16}$$

holds true, then the initial value problem

$$x'(t) = F(x), \tag{17}$$

$$x(0) = \hat{x}, \tag{18}$$

has a unique solution $x \in \mathcal{C}^1([0,1], \mathrm{int}X)$.

Lemma

Let X be a set in $\mathbb{R}^n$, $\hat{x} \in X$, and $f \in \mathcal{C}^2(X, \mathbb{R}^n)$ where Df is regular on X. If $x \in \mathcal{C}^1([0,1], X)$ is a solution to the simplified Newton flow equations

$$x'(t) = -Df\big(x(t)\big)^{-1} f(\hat{x}),\ t \in [0,1], \tag{19}$$

$$x(0) = \hat{x}, \tag{20}$$

then $x(1)$ is a zero of f.

Using previous theorem and lemma, we obtain an improved Newton theorem:

Corollary

Let X be a compact convex set in $\mathbb{R}^n$, and $f \in \mathcal{C}^2(X, \mathbb{R}^n)$ where Df is regular on X. If

$$\hat{x} \in \mathrm{int} X,\ \hat{x} - Df(X)^{-1} f(\hat{x}) \subseteq \mathrm{int} X, \tag{21}$$

then f has a zero in $\mathrm{int} X$.

Condition (21) is equivalent to the disjointness condition

$$[0, f(\hat{x})] \cap \big(f(\hat{x}) + Df(X)(\partial X - \hat{x})\big) = \emptyset, \tag{22}$$

which is good for verification with constraint propagation.

Thank you for listening.