Convex computation of the region of attraction of polynomial control systems

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Region of attraction (ROA)

Given:

- nonlinear control system: $\dot{x} = f(x, u)$
- state constraint set $X \subset \mathbb{R}^n$
- input constraint set $U \subset \mathbb{R}^m$
- final time $T > 0$
- target set $X_T \subset X$

Find ROA X_0 = set of all initial states x_0
that can be steered at time T to the target set X_T
while satisfying the state and input constraints

$$\exists\, u(t) \in U : x(0) \in X_0 \Longrightarrow x(T) \in X_T,\ x(t) \in X,\ \forall t \in [0, T]$$

Our contribution

Under the asssumptions that:

- $f(x,u)$ is a polynomial vector field
- X, X_T and U are compact basic semialgebraic sets

We derive a convex infinite-dimensional linear programming (LP) formulation for ROA computation

We solve this LP with a hierarchy of finite-dimensional semidefinite programming (SDP) relaxations generating guaranteed outer semialgebraic approximations converging almost uniformly to the ROA

Outline

1. Problem statement
2. **Occupation measures**
3. Primal and dual LP
4. LMI relaxations
5. Numerical results

Occupation measures of trajectories

Given a control law $u(t)$ and the resulting state trajectory

$$x(t) = x_0 + \int_0^t f(x(s), u(s))ds$$

define the occupation measure as

$$\mu(A \times B \times C \mid x_0) := \int_0^T I_{A\times B\times C}(t, x(t), u(t))dt$$

for Borel subsets $A \subset [0, T]$, $B \subset X$, $C \subset U$

For any continuous test function $v(t, x, u)$

$$\int_0^T v(t, x(t), u(t))dt = \int_{[0,T]} \int_X \int_U v(t, x, u) d\mu(t, x, u)$$

the occupation measure encodes the trajectory

Measures as initial and final conditions

Now suppose that the initial condition is not a single point x_0 but that its distribution is ruled by a (probability) measure μ_0

We can define accordingly the average occupation measure

$$\mu(A \times B \times C) := \int_X \mu(A \times B \times C \mid x) d\mu_0(x)$$

which measures the average time spent by the trajectory in subsets of $X \times U$

Similarly, the distribution of state at final time is ruled by a (probability) measure

$$\mu_T(B) := \int_X I_B(x(T \mid x)) d\mu_0(x)$$

Liouville's continuity equation

Given $(x_0, u(t))$ and a test function $v \in C^1([0,T] \times X)$, along system trajectories $x(t)$ it holds:

$$\begin{aligned} v(T,x(T)) - v(0,x(0)) &= \int_{[0,T]} \dot{v}(t,x(t))dt \\ &= \int_{[0,T]} \left(\frac{\partial v}{\partial t} + \text{grad}\, v \cdot f\right) dt \\ &= \int_{[0,T]} \int_X \int_U \left(\frac{\partial v}{\partial t} + \text{grad}\, v \cdot f\right) d\mu(t,x,u \,|\, x_0) \end{aligned}$$

Integrating w.r.t. initial measure μ_0 we obtain

$$\int v(T,x)d\mu_T(x) - \int v(0,x)d\mu_0(x) = \int\int\int \left(\frac{\partial v}{\partial t} + \text{grad}\, v \cdot f\right) d\mu(t,x,u)$$

which can be also written (in the sense of distributions) as a linear PDE on measures

$$\frac{\partial \mu}{\partial t} + \text{div}\,(f\mu) = \mu_0 - \mu_T$$

From Cauchy to Liouville

The nonlinear Cauchy ODE

$$\dot{x} = f(x,u), \quad x(0) = x_0$$

has been linearized to a linear Liouville PDE

$$\frac{\partial \mu}{\partial t} + \text{div}\,(f\mu) = \mu_0 - \mu_T$$

in the (infinite-dimensional) Banach space of measures

Nonlinear nonconvex optimal control reduces to
convex linear programming on measures

Compactness, often missing in nonconvex optimal control,
can be retrieved with the weak-star topology

ROA and occupation measures

Dynamics are modeled by Liouville's equation

State and input constraints become measure support constraints

We want to solve the following conic optimization problem

$$
\begin{array}{ll}
\sup & \lambda(\mathrm{spt}\,\mu_0) \\
\text{s.t.} & \frac{\partial \mu}{\partial t} + \mathrm{div}\,(f\mu) = \mu_0 - \mu_T \\
& \mu_0 \geq 0,\ \mathrm{spt}\,\mu_0 \subset X \\
& \mu \geq 0,\quad \mathrm{spt}\,\mu \subset [0,T] \times X \times U \\
& \mu_T \geq 0, \mathrm{spt}\,\mu_T \subset X_T
\end{array}
$$

where λ is the Lebesgue, or uniform, or volume measure

The supremum is w.r.t. unknown measures μ_0, μ, μ_T
The constraints are linear but the objective function is nonlinear

Convex characterization of ROA

Key idea: maximize the mass of μ_0 subject to the constraint that μ_0 is dominated by λ

Linear programming formulation

$$
\begin{array}{rll}
p^* = & \sup & \int \mu_0 \\
& \text{s.t.} & \frac{\partial \mu}{\partial t} + \operatorname{div}(f\mu) = \mu_0 - \mu_T \\
& & \mu_0 + \hat{\mu}_0 = \lambda \\
& & \mu_0 \geq 0,\ \operatorname{spt} \mu_0 \subset X \\
& & \hat{\mu}_0 \geq 0,\ \operatorname{spt} \mu_0 \subset X \\
& & \mu \geq 0,\ \ \operatorname{spt} \mu \subset [0,T] \times X \times U \\
& & \mu_T \geq 0, \operatorname{spt} \mu_T \subset X_T
\end{array}
$$

Theorem: The above supremum is attained by μ_0 equal to the restriction of the Lebesgue measure to the ROA X_0

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5. Numerical results

Dual Banach LP

Using convex duality in Banach spaces, our LP on measures is dual to the following LP on continuous functions

$$
\begin{array}{rlll}
d^* = & \inf & \int w(x)dx & \\
& \text{s.t.} & \frac{\partial v(t,x)}{\partial t} + \operatorname{grad} v(t,x) \cdot f(t,x,u) \leq 0, & \forall (t,x,u) \in [0,T] \times X \times U \\
& & w(x) \geq v(0,x) + 1, & \forall x \in X \\
& & v(T,x) \geq 0, & \forall x \in X_T \\
& & w(x) \geq 0, & \forall x \in X
\end{array}
$$

where the infimum is over $v \in C^1([0,T] \times X)$ and $w \in C(X)$

Theorem: There is no duality gap, i.e. $p^* = d^*$

Outer approximation of ROA

The Hamilton-Jacobi-Bellman constraint

$$\frac{\partial v(t,x)}{\partial t} + \mathrm{grad}\, v(t,x) \cdot f(t,x,u) \leq 0$$

enforces decrease of v along trajectories

and the other constraints imply that the superlevel set

$$X_{0k} := \{x \in X \ : \ w_k(x) \geq 1\}$$

is an outer approximation to X_0 for any feasible w_k

Theorem: There is a sequence of feasible solutions (w_k) converging from above to I_{X_0} in L^1 norm and almost uniformly

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LP on measures

Linear programming (LP) problem

$$\begin{array}{ll} \text{min} & \langle c, m \rangle \\ \text{s.t.} & A(m) = b \\ & m \geq 0 \end{array}$$

where $m = (m_i)$ is a vector of nonnegative Borel measures and

$$\langle c, m \rangle = \sum_i \langle c_i, m_i \rangle = \sum_i \int c_i m_i$$

and

$$A(m) = b \Longleftrightarrow \langle a_j, m \rangle = \sum_i \int a_{ij} m_i = b_j$$

Generalized problem of moments

Several measures m_i supported on semialgebraic sets X_i

All the data are polynomials, so we can replace measures by their moments (e.g. $\int_{X_i} c_i(x) m_i = \int_{X_i} \sum_\alpha c_{i\alpha} x^\alpha m_i = \sum_\alpha c_{i\alpha} \int_{X_i} x^\alpha m_i$)

$$\begin{array}{ll} \min_m & \sum_i \int_{X_i} c_i m_i \\ \text{s.t.} & \sum_i \int_{X_i} a_{ij} m_i = b_j \\ & \text{measures } m_i \end{array} \qquad \begin{array}{ll} \min_y & \sum_i \sum_\alpha c_{i\alpha} y_{i\alpha} \\ \text{s.t.} & \sum_i \sum_\alpha a_{ij\,\alpha} y_{i\alpha} = b_j \\ & \text{moments } y_i \end{array}$$

provided we can handle the representation condition

$$y_{i\alpha} = \int_{X_i} x^\alpha m_i(dx)$$

Moment LP as LMI

Using Putinar's Positivstellensatz we obtain

$$\begin{array}{ll} \min_y & c^T y \\ \text{s.t.} & Ay = b \\ & y_\alpha = \int_X x^\alpha m \\ & X = \{x \,:\, g_k(x) \geq 0, \forall k\} \end{array}$$

infinite-dimensional

LP problem

$$\begin{array}{ll} \min_y & c^T y \\ \text{s.t.} & Ay = b \\ & M_d(y) \succeq 0 \\ & M_d(g_k y) \succeq 0, \forall k \end{array}$$

finite-dim. LMI

relaxation of order d

producing (under some assumption) a converging
hierarchy of finite-dimensional LMI relaxations
that can be solved numerically with SDP solvers

Discretization, analogy with Fourier analysis

Convergence results

To our primal LMI on moments corresponds
a dual LMI on polynomial sum-of-squares (SOS)

Hierarchy of primal-dual LMI relaxations of order $k = 1, 2, \ldots$

Theorem: There is no LMI duality gap, and when $k \to \infty$ the optimum converges to the volume of the ROA

Let w_k denote the SOS polynomial solving the dual LMI of order $k = 1, 2, \ldots$ and let

$$X_{0k} := \{x \in X \ : \ w_k(x) \geq 1\}$$

Theorem: $X_0 \subset X_{0k}$ and $\lim_{k\to\infty} \lambda(X_{0k} - X_0) = 0$

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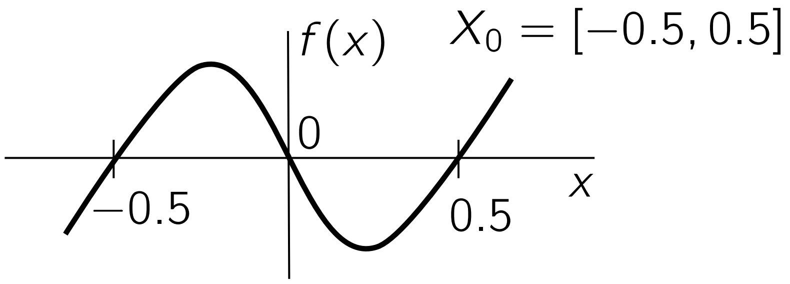
Numerical examples

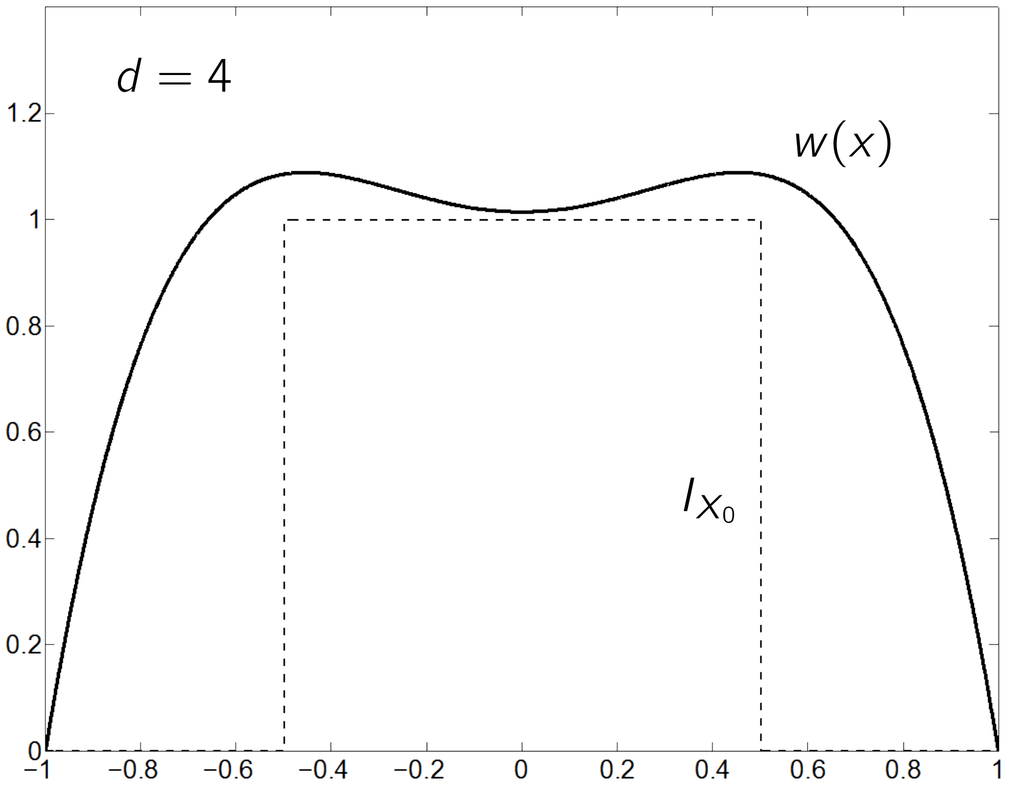
Univariate cubic dynamics

$\dot{x} = x(x - 0.5)(x + 0.5)$

$X = [-1, 1]$

$X_T = [-0.01, 0.01], \quad T = 100$





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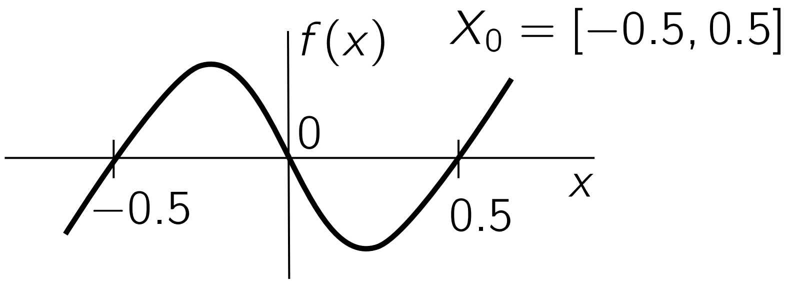

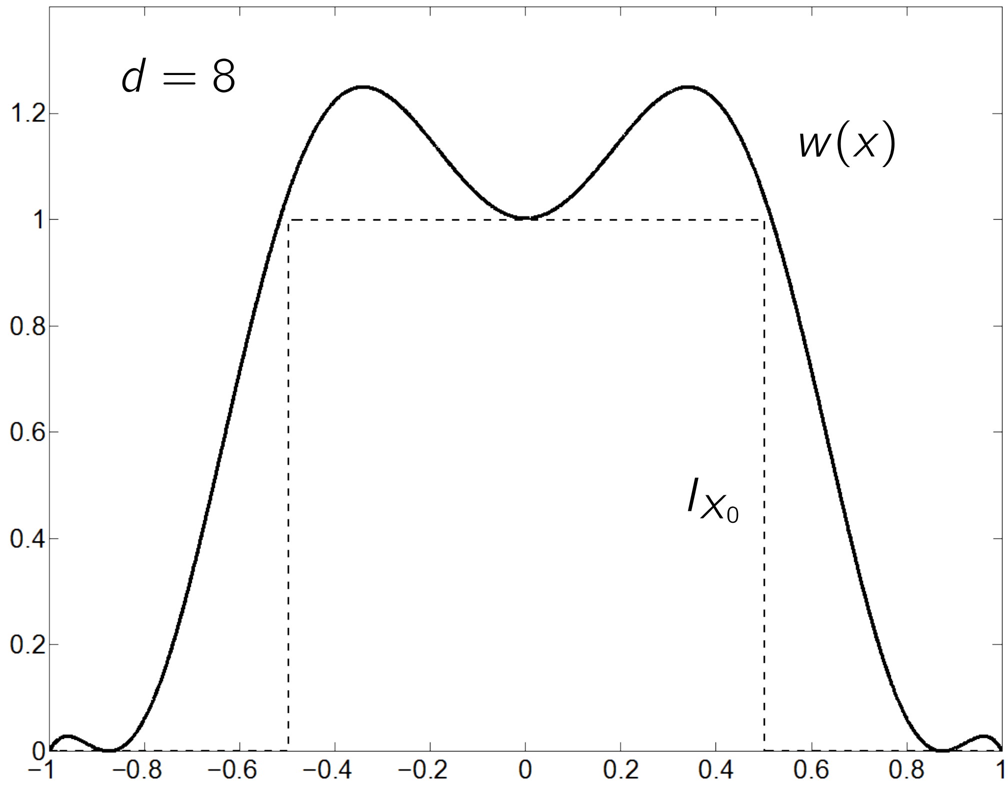

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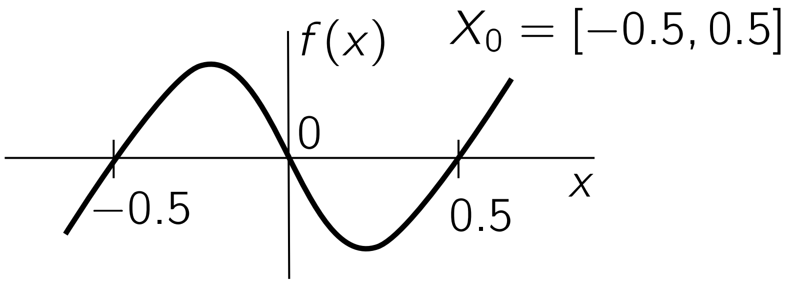

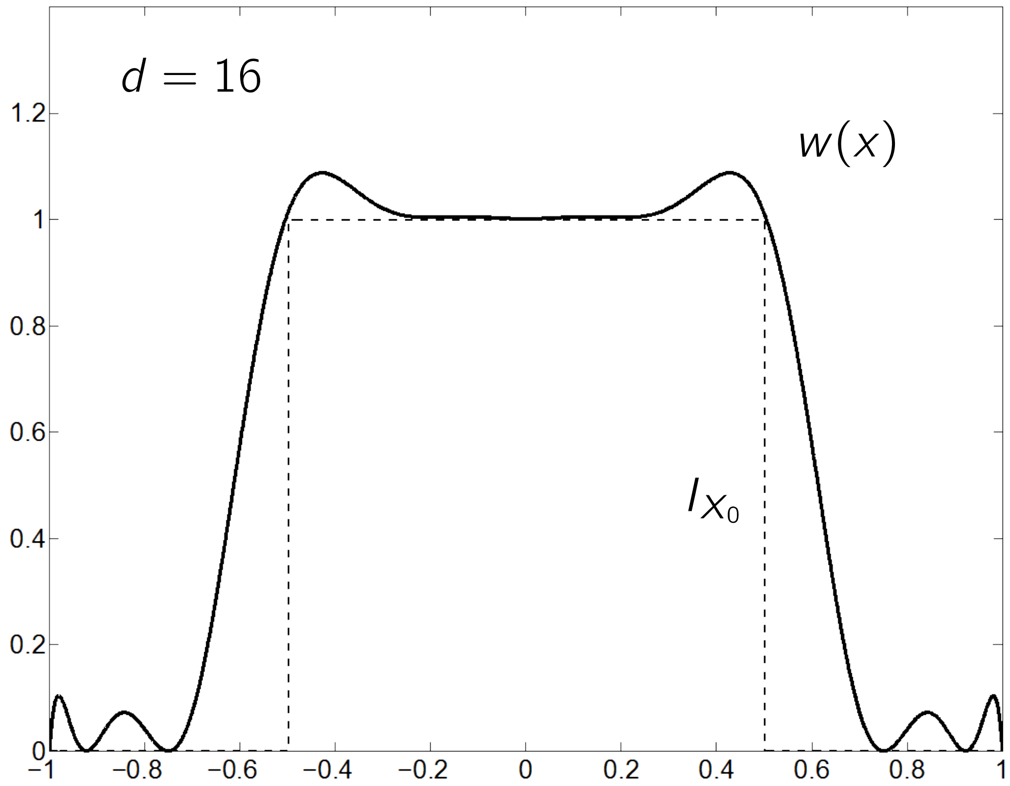

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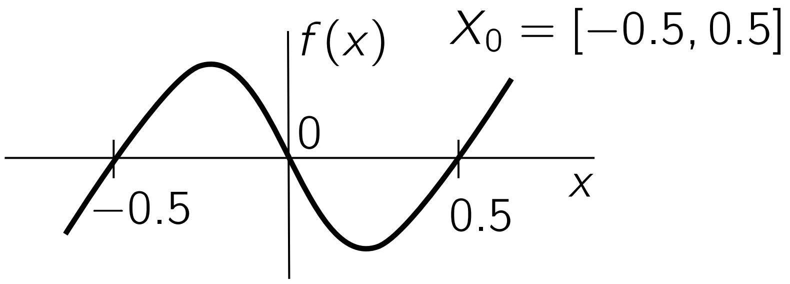
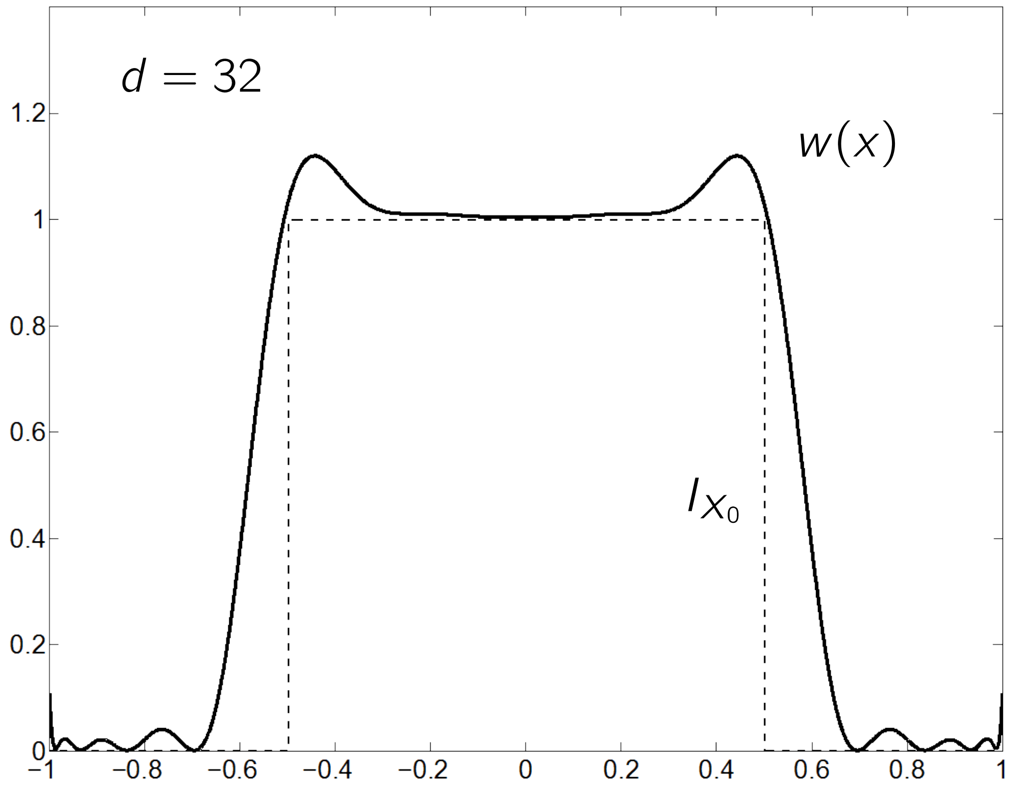




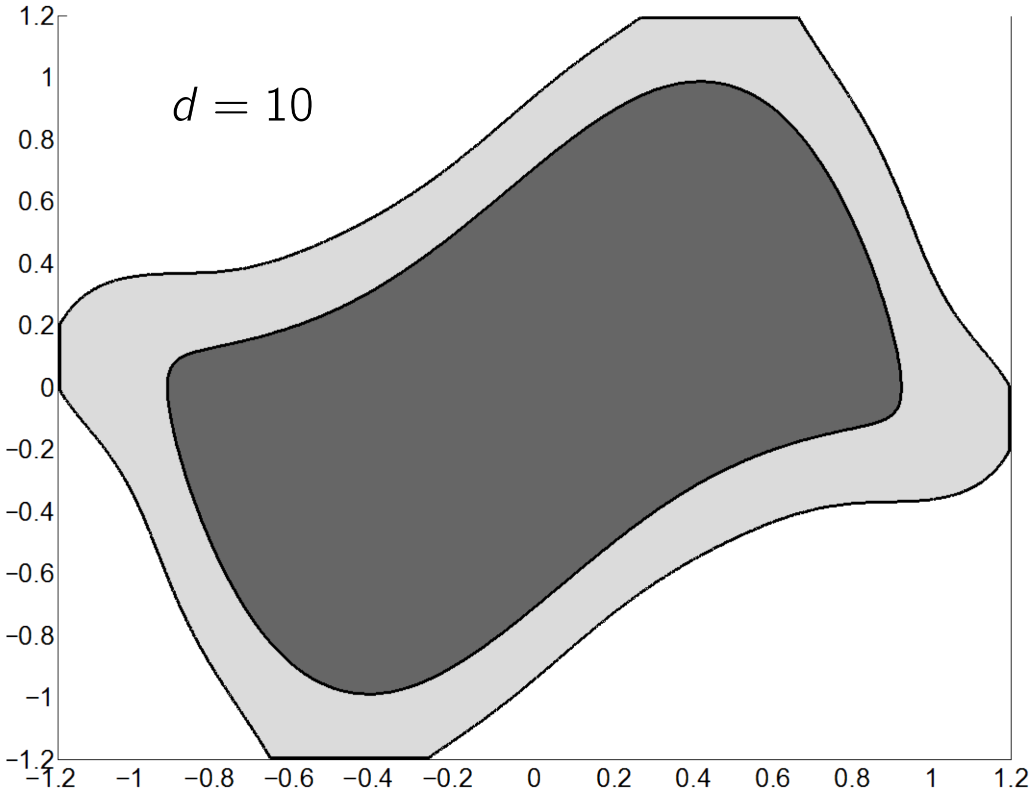
Numerical examples

Backward Van der Pol oscillator

$\dot{x}_1 = -2x_2$
$\dot{x}_2 = 0.8x_1 + 10(x_1^2 - 0.21)x_2$
$X = [-1.2, -1.2]^2$
$X_T = \{x \mid \|x\|_2 \leq 0.01\}$, $T = 100$

Stable equilibrium at the origin with a bounded region of attraction

outer approximations of X_0



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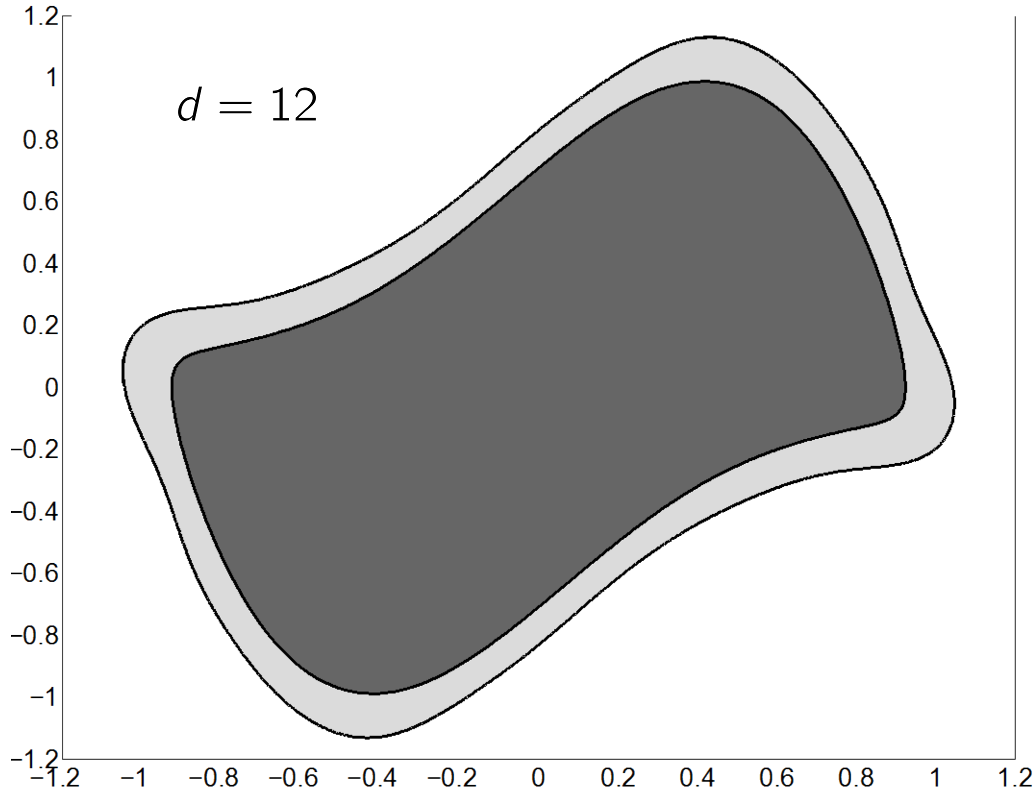

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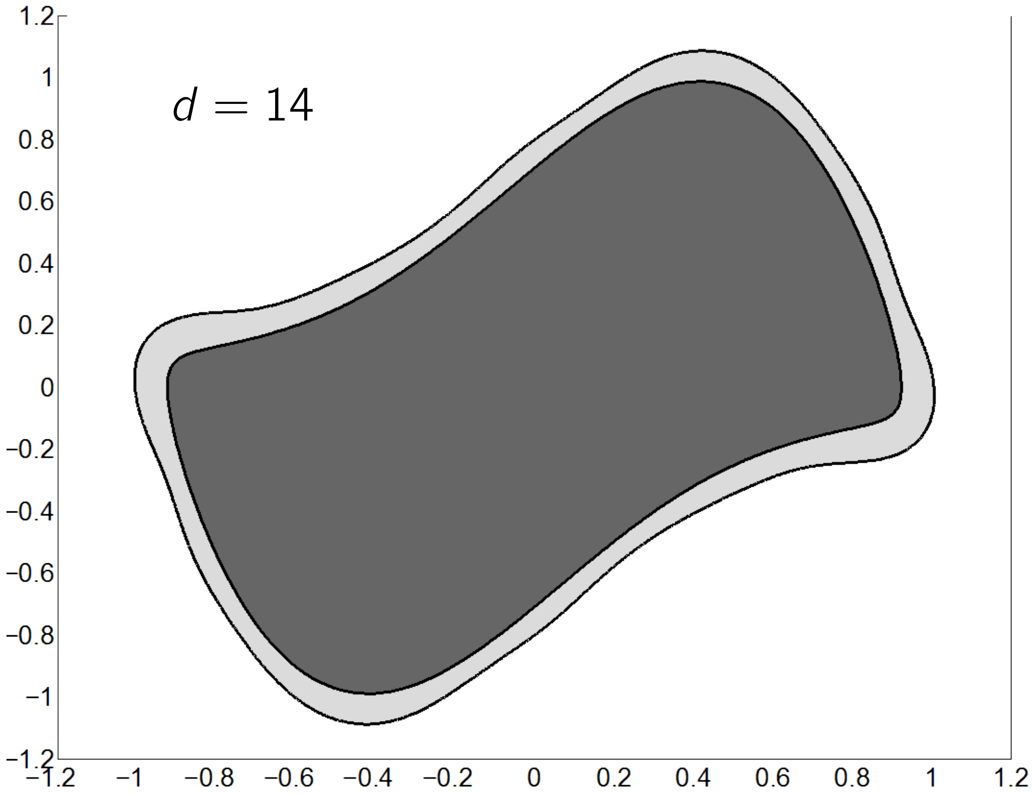

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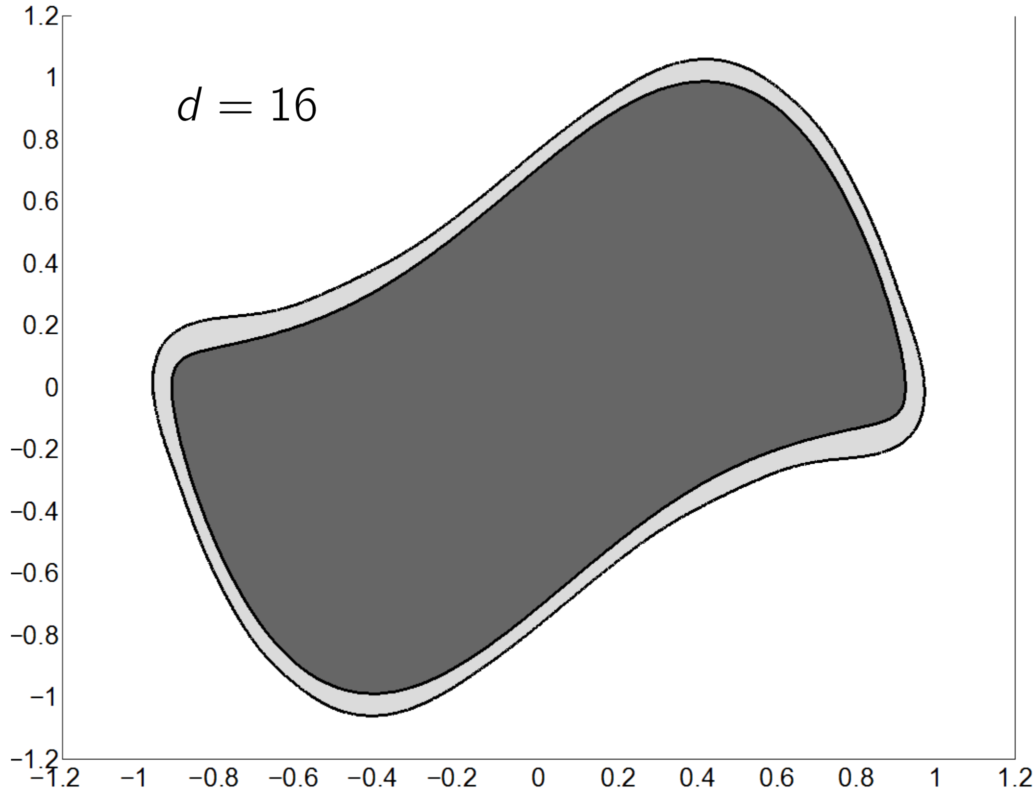

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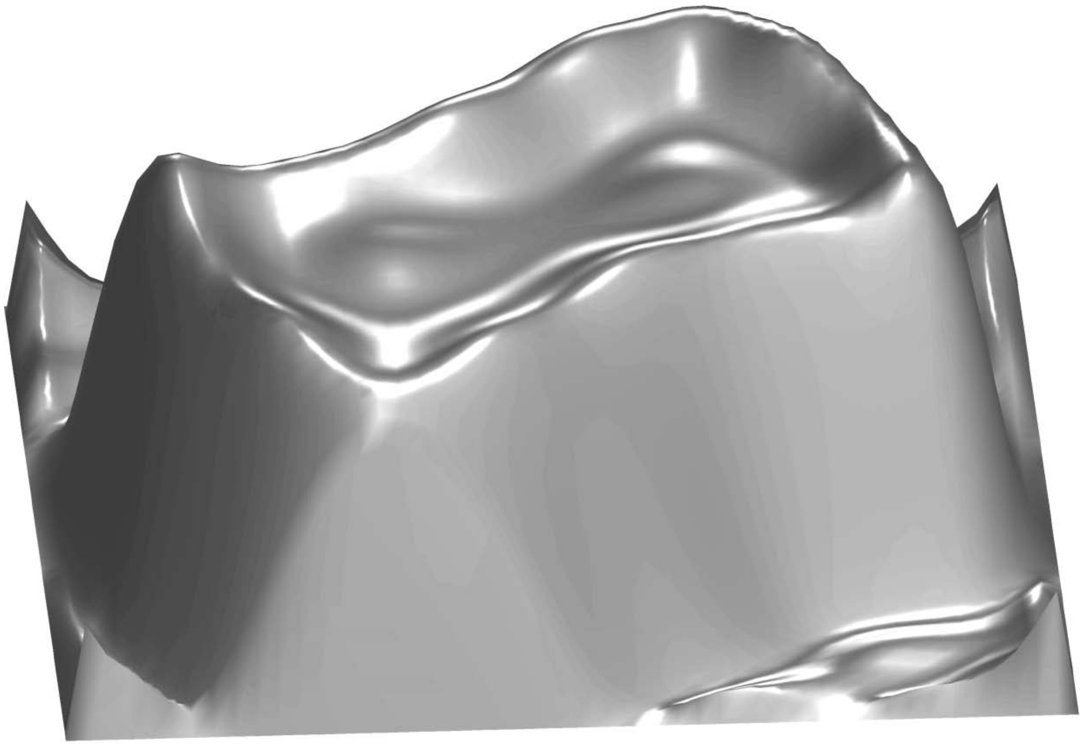
degree 18 approximation to I_{X_0}



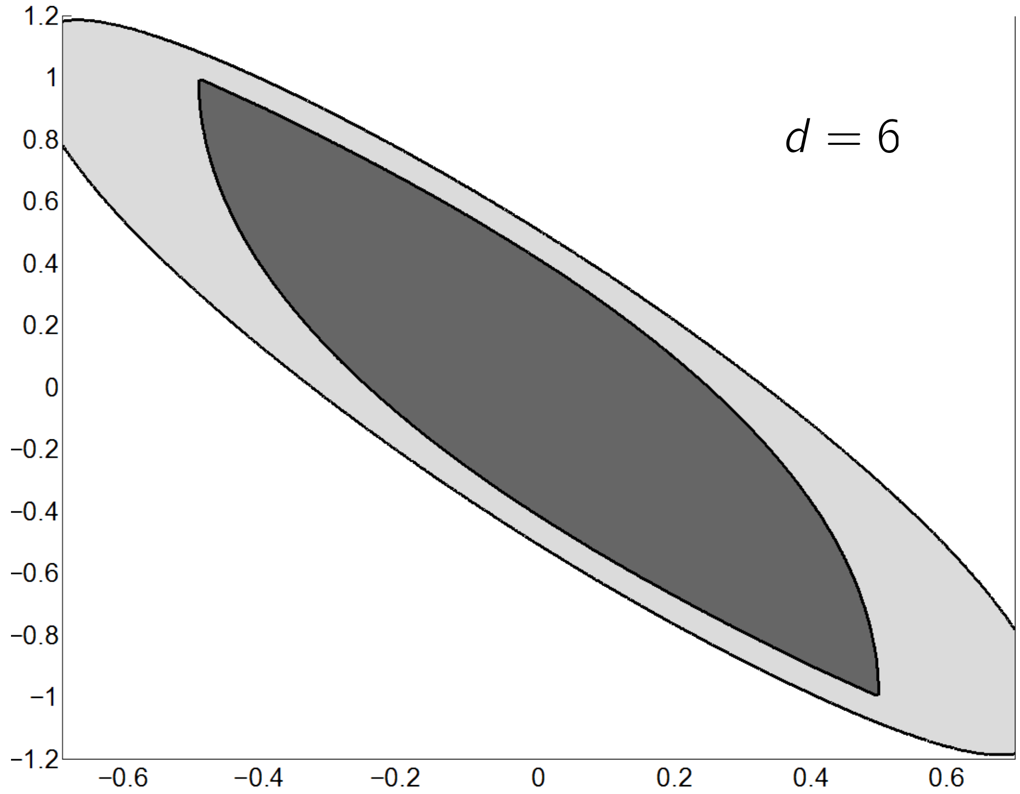
Numerical examples

Double integrator

$\dot{x}_1 = x_2$
$\dot{x}_2 = u$
$X = [-0.7, 0.7] \times [-1.2, 1.2]$
$U = [-1, 1]$
$X_T = \{0\}, \;\; T = 1$

$$X_0 = \{x \,:\, V(x) \le 1\}$$

$$V(x) = \begin{cases} x_2 + 2\sqrt{x_1 + \frac{1}{2}x_2^2} & \text{if} \quad x_1 + \frac{1}{2}x_2|x_2| > 0, \\ -x_2 + 2\sqrt{-x_1 + \frac{1}{2}x_2^2} & \text{otherwise.} \end{cases}$$



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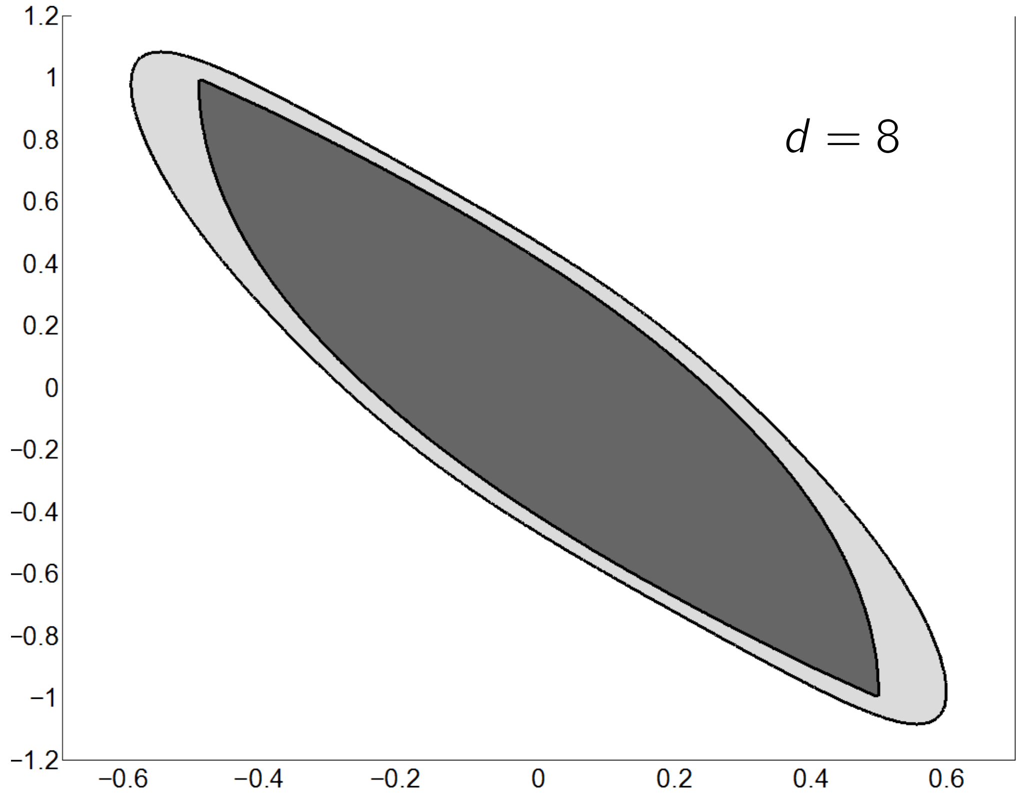

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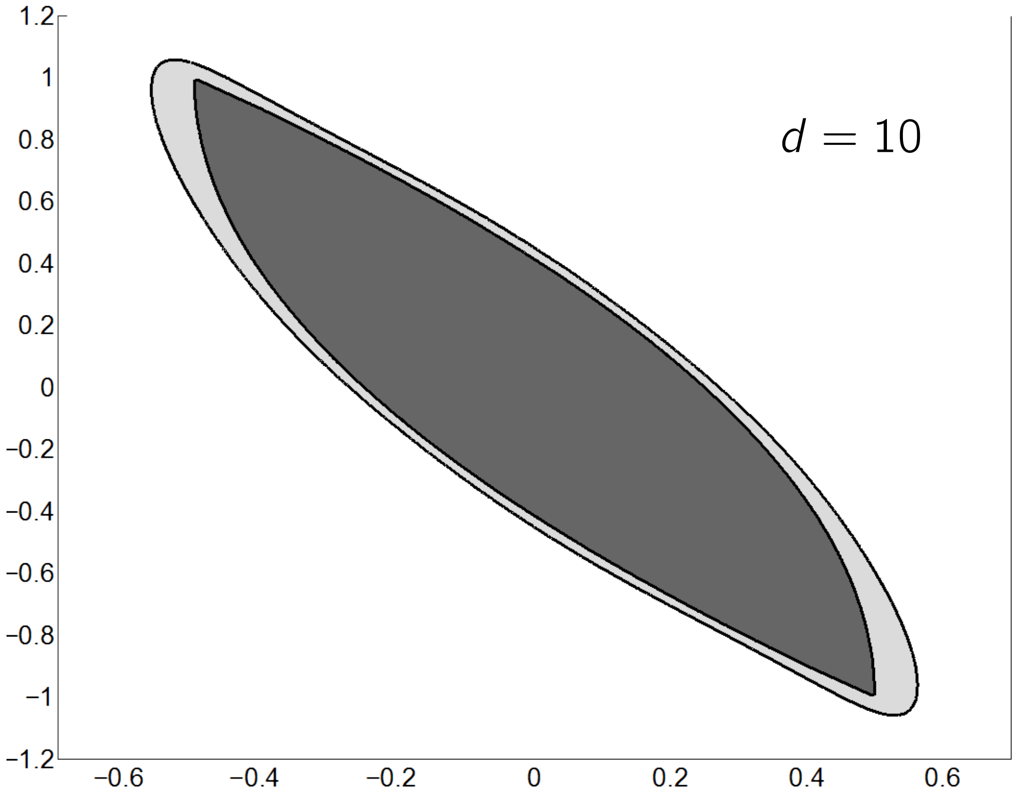

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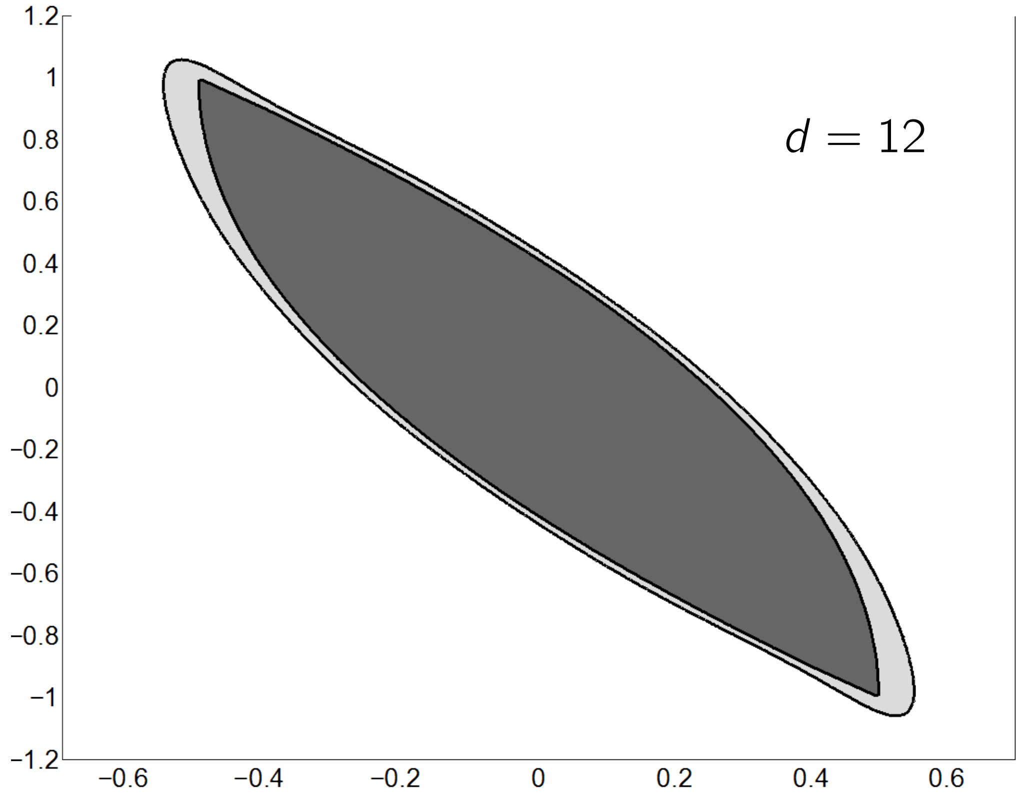

Extensions

ROA computation is convex LP (in Banach space of measures)

For polynomial data, converging hierarchy of LMI relaxations

Outer approximations to ROA with polynomial superlevel sets

Additional properties (e.g. convexity) can be easily enforced

Inner approximations can be also computed with similar ideas

Extensions to reachability, invariance, rational dynamics, piecewise polynomial dynamics, uncertain and stochastic systems

Certifying

Problem data (coefficients of the ODE and constraints) can be assumed rational

Our approach is **numerical** in the sense that we rely on SDP implemented in floating point arithmetic

We cannot reasonably solve LMIs exactly since a solution is typically found in a **huge algebraic extension**

Use multi-precision SDP (SDPA-GMP by M. Kojima's group) ?

Use verified SDP (VSDP by C. Jansson) ?