Life and Death in a Family of Meta-Fibonacci Recursions

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In 1963, as an undergraduate mathematics student, I devised the following curious but natural-seeming recursive definition for an integer-valued function:

$$Q(n) = Q(n - Q(n-1)) + Q(n - Q(n-2))$$

The two initial values I chose were $Q(1) = 1$ and $Q(2) = 1$. Years later, this type of recursion was dubbed by mathematician Steve Tanny a "meta-Fibonacci recursion", because it looks like a nested cousin of the Fibonacci sequence.

The behavior of $Q(n)$ as a function of n turned out to be a strange mixture of regularity and irregularity, alternating between sudden "explosions" and periods of relative calm. Neither I nor anyone else was able to *prove* anything about the Q function — not even that it existed (*i.e.*, was well-defined) for all positive values of n, although computational experimentation showed that it indeed went on and on for millions, even billions of values of n. I was flattered when, sometime in the late 1970s, Paul Erdös fell under the spell of this function and provocatively remarked to me that it was "mathematics not for the twentieth century".

Around the year 2000, my mathematician/physicist friend Greg Huber and I co-invented a family of generalizations of the original sequence $Q(n)$, as follows:

$$Q_{rs}(n) = Q_{rs}(n - Q_{rs}(n-r)) + Q_{rs}(n - Q_{rs}(n-s))$$

where r and s are fixed integers, and where some simple, suitable set of initial values is stipulated, such as {1, 1, 1, 1} or {1, 2, 3, 4}.

Together, Greg and I computationally explored the behavior of this family of functions. We discovered that most of the family members bounced up and down quite chaotically for a while and then suddenly "died" (meaning that at some point, the function grew too big, which meant that the recursion was no longer well-defined, so the sequence could not be extended). How "old" was such a "mortal" meta-Fibonacci function when at last it died? And what was a mortal function's "last word" (*i.e.*, its final value), just before expiring? We carefully explored these two questions as functions of the parameters r and s, and once again, very mysterious behavior cropped up.

There was, in addition, a sparse subfamily whose members, like the original Q sequence, seemed to be *immortal*, with some of them even behaving very unchaotically as a function of n. Which family members were immortal, which immortals were or weren't chaotic, and why?

In this informal, mostly visual talk, I will discuss Greg's and my main findings, using graphs that clearly reveal the puzzling mixture of regularity and irregularity in this rich and challenging family of meta-Fibonacci recursions. Perhaps, now that we are no longer in the twentieth century, someone will at last unravel the mystery of meta-Fibonacci recursions!

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