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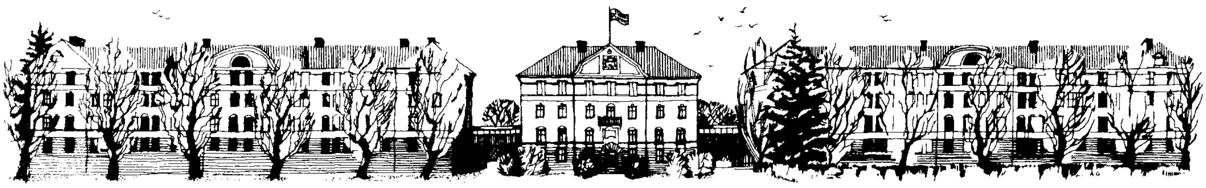

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ULTRAPOWERS AS SHEAVES ON A CATEGORY OF ULTRAFILTERS

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Contents

1. Introduction

In 1993 I. Moerdijk presented a new model of nonstandard arithmetic in the topos of sheaves on a category of filters, $\mathrm{Sh}(\mathbb{F})$. This was later extended by E. Palmgren to a model of nonstandard analysis. The model in particular makes use of the sheaves ${}^*S$, which at any filter $\mathcal{F}$ is the reduced power of the set S over $\mathcal{F}$, ${}^*S(\mathcal{F})$. The details of this will be given in section 1.3. Before this, in section 1.1, we will give a short background to the subject of sheaves and logic and, in section 1.2, some preliminaries.

In this paper we focus our attention on the sheaves on the subcategory of ultrafilters, $\mathrm{Sh}(\mathbb{U})$. The category $\mathbb{U}$ will be discussed in section 2. The sheaves of the form ${}^*S$ now,

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at an ultrafilter $\mathcal{U}$, represents the ultrapower of S over $\mathcal{U}$, ${}^*S(\mathcal{U})$. More details on the sheaves over $\mathbb{U}$ can be found in section 3.

In section 4 we study the internal logic in the topos of sheaves, which is classic since $\mathrm{Sh}(\mathbb{U})$ is an atomic topos. We prove that this logic does not coincide with the logic in any of the ultrapowers ${}^*S(\mathcal{U})$. The category of ultrafilters has a strong connection with ultrafilters under the Rudin-Keisler ordering, for instance we have $\mathcal{U} \leq \mathcal{V}$ if and only if $\mathrm{Hom}_{\mathbb{U}}(\mathcal{V}, \mathcal{U}) \neq \emptyset$. In the paper we define the Rudin-Keisler ordering on $\mathrm{Sh}(\mathbb{U})$ and study the consequences of it in our setting.

In the paper we investigate the properties of $\mathrm{Sh}(\mathbb{U})$. We establish two transfer principles: external transfer, which is corresponding to Łoś theorem, and an internal transfer principle. We show that the topos theoretic axiom of choice does not hold in $\mathrm{Sh}(\mathbb{U})$ but establish some weak form of it and also prove some other properties similar to results proved by Palmgren about $\mathrm{Sh}(\mathbb{F})$.

In section 5 we show that the topos can be used to model Nelson's internal set theory (IST). IST is an axiomatic approach to nonstandard analysis, which adds to ZFC a undefined unary predicate $\mathrm{St}(x)$, for the standard sets, and axioms relating the standard and nonstandard sets.

1.1. **History.** This is a short history of sheaves and topoi, with special emphasis on their use in logic. For a more thorough history, see S. Mac Lane and I. Moerdijk [8].

A. Grothendieck defined the first topos in the early sixties. This was a collection of sheaves on a site, and was inspired by his study of cohomology for generalized spaces. Sheaves, or the idea behind sheaves, had been known, at least, as early as in the 19th century. The first general and explicit definition of a sheaf on a space was published by J. Leray in 1945. The sheaf was defined in terms of the closed sets of that space. H. Cartan rephrased the definition in terms of open sets in his seminars 1948-49 and 1950-51.

Using sheaves of abelian groups on a topological space it became possible to define cohomology of a topological space with variable coefficients. A sheaf A of abelian groups on a topological space X is a family of abelian groups A_x, parametrized by the points $x \in X$ in a nice way such that it respects some locality and integrality properties. J.P. Serre and others realized that sheaves could be used also in algebraic geometry by defining sheaves not only on subsets U of X, but on any mapping $U \to X$.

Then Grothendieck gave a more general definition of sheaves by replacing the partially ordered collection of open subsets of a space by objects in a category $\mathbf{C}$, in which some suitable families of maps $U_i \to X$ (for $i \in I$) form "covers" of objects X in $\mathbf{C}$. For such a "Grothendieck topology" a sheaf is a functor which respects the topology on $\mathbf{C}$ in a appropriate manner. The sheaves on a category $\mathbf{C}$ with a Grothendieck topology J (a "site") forms a "Grothendieck topos" $\mathrm{Sh}(\mathbf{C}, J)$.

On another note, F.W. Lawvere in 1963 embarked on the grand project of a purely categorical foundation for all mathematics, starting with an appropriate axiomatization of the category of sets, replacing membership by composition of functions. When Lawvere learned of the Grothendieck topoi, he soon observed that such a topos admits the basic operations of set theory, such as the formation of the exponential set Y^X (of all functions from X to Y) and the power set $P(X)$ (of all subsets of X).

At about the same time M. Tierney saw that Grothendieck's work could lead to an axiomatic study of sheaves. Working together, Lawvere and Tierney discovered an axiomatization of categories of sheaves of sets (and, in particular, of the category of sets) via an appropriate formulation of the set-theoretic properties. This was the definition of the "elementary topos", which is defined without any set-theoretic assumptions. Any Grothendieck topos is an elementary topos, but not conversely.

A second input into the use of topoi in logic was from the "forcing" method used by P. Cohen to prove the independence of the continuum hypothesis (CH), and other set-theoretic axioms, from the rest of Zermelo-Fraenkel set theory (ZF). The method works by expanding a model of ZF to a new model by forcing some sets to exist in the new model, in the case of CH a subset $B \subset \mathbb{R}$ such that the cardinality of B is strictly between the cardinalities of $\mathbb{N}$ and $\mathbb{R}$.

This forcing technique was later rephrased by R. Solovay and D. Scott in terms of Boolean-valued models, where the truth-values are taken from an arbitrary Boolean algebra. Shortly after this, Lawvere and Tierney discovered that the Cohen forcing could be explained in terms of topoi: indeed, using Cohen's construction one obtains a topos with the desired properties.

1.2. **Preliminaries.** We will assume knowledge of the basic categorical devices (pullbacks, functors etc.) and will give some basic definitions and theorems regarding sheaves and topoi. For a more detailed presentation, see Mac Lane and Moerdijk [8] or P. Johnstone [6].

Let $\mathbf{C}$ be a category. A *sieve* S on an object C in $\mathbf{C}$ is a family of morphisms in $\mathbf{C}$, all with codomain C, such that $f \in S \Rightarrow f \circ g \in S$, whenever this composition makes sense. If S is a sieve on C and $h : D \to C$ is any arrow to C, then $h^*(S) = \{g \mid \mathrm{cod}(g) = D, h \circ g \in S\}$ is a sieve on D.

Definition 1.1. A *Grothendieck topology* on a category $\mathbf{C}$ is a function J which assigns to each object C in $\mathbf{C}$ a collection $J(C)$ of sieves on C in such a way that

(i) the maximal sieve $t_C = \{f \mid \mathrm{cod}(f) = C\}$ is in $J(C)$,
(ii) (stability) if $S \in J(C)$, then $h^*(S) \in J(D)$ for any arrow $h : D \to C$,
(iii) (transitivity) if $S \in J(C)$ and R is any sieve on C such that $h^*(R) \in J(D)$ for all $h : D \to C$ in S, then $R \in J(C)$.

A *site* is a pair $(\mathbf{C}, J)$, with a category $\mathbf{C}$ and a Grothendieck topology J on it. If $S \in J(C)$ one says that S *covers* C (or that S is a *covering sieve*).

An example of a site is a topological space X with the usual notion of a cover: The topology $\mathcal{O}(X)$ of X is partially ordered under inclusion. The set $(\mathcal{O}(X), \subseteq)$ can be viewed as a category, with objects the open sets U of X, and exactly one morphism $U \to V$ if and only if $U \subseteq V$. A sieve S on U is now a family of open subsets of U with the property that $V' \subseteq V \in S$ implies $V' \in S$. A Grothendieck topology on X is given by saying that S covers U if and only if U is contained in the union of the sets in S.

Usually on a topological space one describes a cover of U as just a family $\{U_i \mid i \in I\}$ of open subsets of U such that $U = \bigcup U_i$. Such a family is not necessarily a sieve, but it can be used to generate a sieve - namely the collection of all open $V \subseteq U$ such that $V \subseteq U_i$, for some $i \in I$.

In a similar way for an arbitrary category $\mathbf{C}$ it is often enough to consider a *basis* (for a Grothendieck topology) and then use this basis to generate the covering sieves.

Now for the sheaves. Let $(\mathbf{C}, J)$ be a site. A *presheaf* (of sets) on $\mathbf{C}$ is a contravariant functor $P : \mathbf{C} \to \mathbf{Sets}$, i.e. a functor $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$. If P is a presheaf and the sieve S covers an object C in $\mathbf{C}$, then a *matching family* for S of elements of P is a function which assigns to each element $f : D \to C$ in S an element $x_f \in P(D)$, in such a way that $P(g)(x_f) = x_{f\circ g}$, for all $g : E \to D$ in $\mathbf{C}$. An *amalgamation* of such a matching family is an element $x \in P(C)$ such that $P(f)(x) = x_f$, for all $f \in S$.

We can now give the definition of a sheaf:

Definition 1.2. A presheaf P on a site $(\mathbf{C}, J)$ is a *sheaf* if and only if every matching family for any cover of any object of $\mathbf{C}$ has a unique amalgamation.

This definition can be simplified somewhat by expressing it in terms of a basis for the topology J.

An example of a sheaf on a topological space X (considered as a site as above) is the functor taking any open subset U to the set of continuous real-valued functions on U. This functor has the following properties

(i) If $f : U \to \mathbb{R}$ is continuous and $V \subseteq U$, then f restricted to V, $f|_V : V \to \mathbb{R}$, is continuous.

(ii) If U is covered by open subsets U_i (with $i \in I$), and there are continuous functions $f_i : U_i \to \mathbb{R}$ such that for every $i, j \in I$ we have $f_i(x) = f_j(x)$, for $x \in U_i \cap U_j$, then there is an unique function $f : U \to \mathbb{R}$ such that $f|U_i = f_i$.

Property (i) shows that the functor is a presheaf, while (ii) is the sheaf-condition ,with $U_i \mapsto f_i$ as the matching family and f as the unique amalgamation ((ii) is a variation on the matching condition, for the case when the collection of U_i:s is only a basis and not a sieve).

All of the above is done only for sheaves of sets. Other types of sheaves are achieved by replacing the category **Sets** with some suitable other category.

The category of sheaves on a site $(\mathbf{C}, J)$ with natural transformations as morphisms is called a *Grothendieck topos*, $\mathrm{Sh}(\mathbf{C}, J)$. A Grothendieck topos has some very nice properties. Many of these are expressed by saying that it is an elementary topos (we will often just say topos):

Definition 1.3. A category $\mathcal{E}$ is an *(elementary) topos* if:

(i) it has all finite limits,

and is equipped with

(ii) an object Ω,

(iii) a function P, which assigns to each object B of $\mathcal{E}$ an object PB of $\mathcal{E}$,

(iv) for each object A of $\mathcal{E}$ two isomorphisms, each natural in A,

$$\mathrm{Sub}_{\mathcal{E}}A \cong \mathrm{Hom}_{\mathcal{E}}(A, \Omega), \tag{1}$$

and

$$\mathrm{Hom}_{\mathcal{E}}(B \times A, \Omega) \cong \mathrm{Hom}_{\mathcal{E}}(A, PB). \tag{2}$$

To be natural in A means that the isomorphisms are functorial with respect to A. $\mathrm{Sub}_{\mathcal{E}}A$ in (1) is the collection of subobjects of A, i.e. monomorphisms into A.

In the definition the object Ω in (ii) is the *subobject classifier* and (1) shows that the subobjects of an object A is internally represented as morphisms from A to Ω. The function P in (iii) will serve as the power set operator, and from (2) it follows that PB is the exponential Ω^B.

The perhaps simplest example of a topos is the category **Sets**. Here the subobject classifier Ω is the set $\{0, 1\}$. Equation (1) says that every subset B of a set A can be represented as a function from A to $\{0, 1\}$ ($\chi_B(x) = 1$, if $x \in B$). If you put $A = \{*\}$, the one-point set, in equation (2) then you get that functions from B to $\{0, 1\}$ (i.e. the subsets of B) can be identified with elements in PB, i.e. PB is the powerset of B.

Every topos is cartesian closed since the existence of arbitrary exponentials A^B can be proved from the existence of the exponentials on the form Ω^B.

We now use the topoi for mathematical logic. Assume that we have a many-sorted language L and that we want to interpret the theories over L in a topos $\mathcal{E}$. First we associate to each sort in the language an object in $\mathcal{E}$. Formulas are interpreted as subobjects of the product of the objects of the sorts of the free variables of the formula. As an example, assume that $\Theta(x, y)$ is a formula with x of sort X and y of sort Y as its only free variables. Assume that X is interpreted in $\mathcal{E}$ by S and Y by T. Then $\Theta(x, y)$ is interpreted as a subobject, $M_{x,y}(\Theta(x, y))$ of $S \times T$, i.e. a monomorphism $m : M_{x,y}(\Theta(x, y)) \to S \times T$. This is called the Mitchell-Bénabou language for $\mathcal{E}$. It is based on the observation that the topos behaves like a "universe of sets".

Now the truth values for a formula are given internally by the subobject classifier, which in general is not a Boolean but a Heyting algebra. This means that we do not get the same object back after double negation, and thus the internal logic in a topos is, in general, intuitionistic. This semantic is called the "Kripke-Joyal semantics" for a topos, and it is a generalization of the semantics for intuitionistic logic presented by S. Kripke.

We will give the semantics for a Grothendieck topos (the *sheaf semantics*). This is a simplification of the semantics in a general topos. Given a sheaf X, an object C, $\alpha \in X(C)$ and a morphism $f : D \to C$ we will use $\alpha \cdot f$ as an abbreviation for $X(f)(\alpha)$.

Theorem 1.4. *For a Grothendieck topology J on $\mathbf{C}$, let X be a sheaf on $\mathbf{C}$, while $\phi(x)$, $\psi(x)$ are formulas in the language of the topos $\mathrm{Sh}(\mathbf{C}, J)$ and x is a free variable of sort X. Let C be an object in $\mathbf{C}$ and let $\alpha \in X(C)$.*
Then

(i) $C \Vdash \phi(\alpha) \wedge \psi(\alpha)$ *if and only if* $C \Vdash \phi(\alpha)$ *and* $C \Vdash \psi(\alpha)$,
(ii) $C \Vdash \phi(\alpha) \vee \psi(\alpha)$ *if and only if there is a covering* $\{f_i : C_i \to C\}$ *such that for each index i, either* $C_i \Vdash \phi(\alpha \cdot f_i)$ *or* $C_i \Vdash \psi(\alpha \cdot f_i)$,
(iii) $C \Vdash \phi(\alpha) \to \psi(\alpha)$ *if and only if for all* $f : D \to C$, $D \Vdash \phi(\alpha \cdot f)$ *implies* $D \Vdash \psi(\alpha \cdot f)$,
(iv) $C \Vdash \neg\phi(\alpha)$ *if and only if for all* $f : D \to C$, *if* $D \Vdash \phi(\alpha \cdot f)$ *then the empty family is a cover of D.*

Moreover, if $\phi(x, y)$ is a formula with free variables x and y of sorts X and Y, then for $\alpha \in X(C)$,

(v) $C \Vdash \exists y \phi(\alpha, y)$ *if and only if there are a covering* $\{f_i : C_i \to C\}$ *and elements* $\{\beta_i \in Y(C_i)\}$ *such that* $C_i \Vdash \phi(\alpha \cdot f_i, \beta_i)$ *for each index i,*

(vi) *$C \Vdash \forall y\phi(\alpha, y)$ if and only if for all $f : D \to C$ and all $\beta \in Y(D)$ one has $D \Vdash \phi(\alpha \cdot f, \beta)$.*

The forcing relation "$C \Vdash$" has two additional properties:

(i) (Monotonicity) If $C \Vdash \phi(\alpha)$ and $f : D \to C$ then $D \Vdash \phi(\alpha \cdot f)$.
(ii) (Local character) If $\{f_i : C_i \to C\}$ is a cover in J such that $C_i \Vdash \phi(\alpha \cdot f_i)$ for all i, then $C \Vdash \phi(\alpha)$.

1.3. **Reduced powers as sheaves.** In 1993 Moerdijk [9] introduced a model of constructive nonstandard arithmetic in the sheaves on a category of filters $\mathbb{F}$. Further contributions to this model where made by E. Palmgren [12, 13, 14, 15].

Definition 1.5. The category $\mathbb{F}$ has as *objects* pairs $(A, \mathcal{F})$, where A is a set and $\mathcal{F}$ a filter on A. The *morphisms* $\alpha : (A, \mathcal{F}) \to (B, \mathcal{G})$ are equivalence classes of partial functions $\alpha : A \to B$ such that

(i) α is defined on some $F \in \mathcal{F}$,
(ii) $\alpha^{-1}(G) \in \mathcal{F}$, for all $G \in \mathcal{G}$.

Two such partial functions $\alpha : F \to B$ and $\alpha' : F' \to B$ are equivalent if there is $E \subseteq F \cap F'$ such that $\alpha|_E = \alpha'|_E$.

This category of filters $\mathbb{F}$ was introduced by V. Koubek and J. Reiterman [7] and studied further by A. Blass [1]. Initial object 0 in $\mathbb{F}$ is $(\emptyset, \{\emptyset\})$.

Proposition 1.6. *For morphisms $\alpha : (A, \mathcal{F}) \to (B, \mathcal{G})$ we have:*

(i) *α is mono if and only if there is a $F \in \mathcal{F}$ such that α is injective on F,*
(ii) *α is epi if and only if $\alpha(\mathcal{F}) = \mathcal{G}$.*

Moerdijk defined a Grothendieck topology on $\mathbb{F}$ as follows:

Definition 1.7. A finite family $\{\beta_i : \mathcal{G}_i \to \mathcal{F}\}_{i=1}^n$ is a *covering* if for any $G_1 \in \mathcal{G}_1, \ldots, G_n \in \mathcal{G}_n$ there is a $F \in \mathcal{F}$ such that

$$\beta_1(G_1) \cup \ldots \cup \beta_n(G_n) \supseteq F.$$

Over the resulting topos he studied, particularly, the representable sheaves of the form ${}^*S = \mathrm{Hom}_{\mathbb{F}}(-, (S, \{S\}))$. At any filter $\mathcal{F}$, ${}^*S(\mathcal{F})$ is the reduced power of S over $\mathcal{F}$. Thus, if you restrict the underlying category to the full subcategory of ultrafilters, $\mathbb{U}$, you can study ultrapowers as sheaves.

In this way the ordinary construction of an ultrapower can be seen as a sheaf on the ultrafilters on a set I. Now the ultrafilters on I can be thought of as the Stone space of the Boolean algebra of the subsets of I. This way of thinking about ultrapowers (or in general ultraproducts) has been generalized by D.P. Ellerman [5] to sheaves on the Stone space of the complete Boolean algebra of the regular open sets of any space I. In this setting you get as sheaves generalized ultraproducts called ultrastalks.

The sheaf semantics take the following form in the topos of sheaves on $\mathbb{F}$:

Theorem 1.8. *Let $\mathcal{F}$ be a filter, and let Φ, Ψ be any formulas. If $\alpha_1 \in X_1(\mathcal{F}), \ldots, \alpha_n \in X_n(\mathcal{F})$ we write $\overline{\alpha} = \alpha_1, \ldots, \alpha_n$. Notice that if $\alpha \in {}^*S(\mathcal{F})$ then $\alpha \cdot \beta = \alpha \circ \beta$. Then*

(i) *$\mathcal{F} \Vdash \Phi(\overline{\alpha}) \wedge \Psi(\overline{\alpha})$ if and only if $\mathcal{F} \Vdash \Phi(\overline{\alpha})$ and $\mathcal{F} \Vdash \Psi(\overline{\alpha})$,*

(ii) $\mathcal{F} \Vdash \Phi(\overline{\alpha}) \vee \Psi(\overline{\alpha})$ *if and only if for some cover* $\{\beta_i : \mathcal{G}_i \to \mathcal{F}\}_{i=1}^n$ *at least one of* $\mathcal{G}_i \Vdash \Phi(\overline{\alpha} \cdot \beta_i)$ *or* $\mathcal{G}_i \Vdash \Psi(\overline{\alpha} \cdot \beta_i)$ *holds for each* $i = 1, \dots, n$,

(iii) $\mathcal{F} \Vdash \Phi(\overline{\alpha}) \to \Psi(\overline{\alpha})$ *if and only if for all* $\beta : \mathcal{G} \to \mathcal{F} : \mathcal{G} \Vdash \Phi(\overline{\alpha} \cdot \beta)$ *implies* $\mathcal{G} \Vdash \Psi(\overline{\alpha} \cdot \beta)$,

(iv) $\mathcal{F} \Vdash (\exists x \in {}^*S)\Phi(\overline{\alpha}, x)$ *if and only if there is a cover* $\{\beta_i : \mathcal{G}_i \to \mathcal{F}\}_{i=1}^n$ *and a set of witnesses* $\{\delta_i \in {}^*S(\mathcal{G}_i)\}_{i=1}^n$ *such that* $\mathcal{G}_i \Vdash \Phi(\overline{\alpha} \cdot \beta_i, \delta_i)$ *for all* $i = 1, \dots, n$,

(v) $\mathcal{F} \Vdash (\forall x \in {}^*S)\Phi(\overline{\alpha}, y)$ *if and only if for all* $\beta : \mathcal{G} \to \mathcal{F}$ *and* $\delta \in {}^*S(\mathcal{G})$: $\mathcal{G} \Vdash \Phi(\overline{\alpha} \cdot \beta, \delta)$.

We also define what it means to be standard for a $\gamma \in {}^*S(\mathcal{F})$:

(vi) $\mathcal{F} \Vdash \mathrm{St}(\gamma)$ *if and only if there is a cover* $\{\beta_i : \mathcal{G}_i \to \mathcal{F}\}_{i=1}^n$ *and some elements* $s_1, \dots, s_n \in S$, *such that for each* i, $\gamma \circ \beta_i = s_i$ *on some* $G \in \mathcal{G}_i$.

For the ultrafilters in $\mathbb{F}$ we have the following result from Palmgren [13]:

Theorem 1.9.

(i) *Any morphism from a proper filter to an ultrafilter is a covering map.*

(ii) *Any cover of an ultrafilter contains a single map covering the ultrafilter.*

These theorems, together with the local character of the forcing relation, give that the forcing relation takes a much nicer form at an ultrafilter $\mathcal{U}$ in $\mathbb{F}$:

Corollary 1.10. *Let* $\mathcal{U}$ *be an ultrafilter, and let* Φ, Ψ *be any formulas. Then*

(i) $\mathcal{U} \Vdash \mathrm{St}(\gamma)$ *if and only if* γ *is constant on some* $U \in \mathcal{U}$,

(ii) $\mathcal{U} \Vdash \Phi(\overline{\alpha}) \vee \Psi(\overline{\alpha})$ *if and only if* $\mathcal{U} \Vdash \Phi(\overline{\alpha})$ *or* $\mathcal{U} \Vdash \Psi(\overline{\alpha})$,

(iii) $\mathcal{U} \Vdash \Phi(\overline{\alpha}) \to \Psi(\overline{\alpha})$ *if and only if* $\mathcal{U} \Vdash \Phi(\overline{\alpha})$ *implies* $\mathcal{U} \Vdash \Psi(\overline{\alpha})$,

(iv) $\mathcal{U} \Vdash (\exists^{st} y \in {}^*S)\Phi(\overline{\alpha}, y)$ *if and only if for some* $s \in S$, $\mathcal{U} \Vdash \Phi(\overline{\alpha}, {}^*s)$.

2. The category of ultrafilters $\mathbb{U}$

2.1. **$\mathbb{U}$ as a subcategory of $\mathbb{F}$.** Let $\mathbb{U}$ be the full subcategory of ultrafilters in $\mathbb{F}$. In this section we will introduce $\mathbb{U}$ in terms of the category $\mathbb{F}$. If $(A, \mathcal{F})$ is a filter and $B \subseteq A$, we will use the notation $\mathcal{F} \restriction B$ for the filter $\{F \cap B \mid F \in \mathcal{F}\}$ on B.

Proposition 2.1. *A filter* $(A, \mathcal{U})$ *in* $\mathbb{F}$ *is an ultrafilter if and only if* $(A, \mathcal{U}) \not\cong 0$ *and for any covering* $\{\beta_i : \mathcal{F}_i \to \mathcal{U}\}_{i=1}^n$ *there is an* i *such that* $\beta_i : \mathcal{F}_i \to \mathcal{U}$ *is a covering.*

Proof. "$\Longrightarrow$": Theorem 1.9(ii).

"$\Longleftarrow$": Take $(A, \mathcal{U})$ as in the proposition. Take $B_1 \subseteq A$ and $B_2 = A \setminus B_1$. From the definition of the topology on $\mathbb{F}$ follows that $\{\mathcal{U} \restriction B_i \hookrightarrow \mathcal{U}\}_{i=1}^2$ is a covering. From Theorem 1.9 one of the maps is a covering, say $\mathcal{U} \restriction B_1 \hookrightarrow \mathcal{U}$. But $B_1 \in \mathcal{U} \restriction B_1$ and hence $B_1 \in \mathcal{U}$. □

Recall that a family of objects $\mathcal{C}$ in a category $\mathbf{C}$ is said to *generate* $\mathbf{C}$ if, for every pair $f \neq g : A \to B$ in $\mathbf{C}$, there is an object $C \in \mathcal{C}$ and morphism $h : C \to A$ such that $f \circ h \neq g \circ h$.

Proposition 2.2. *The collection of ultrafilters in* $\mathbb{F}$ *generates* $\mathbb{F}$.

Proof. Take any two morphisms $\alpha, \beta : (A, \mathcal{F}) \to (B, \mathcal{G})$ in $\mathbb{F}$ such that $\alpha \neq \beta$. Then, for every $F \in \mathcal{F}$ there is a $x \in F$ such that $\alpha(x) \neq \beta(x)$. Let $A' = \{x \in A \mid \alpha(x) \neq \beta(x)\}$. Then $\mathcal{F} \restriction A'$ is nontrivial. Let $\mathcal{U}$ be an ultrafilter expansion of $\mathcal{F} \restriction A'$ and $\iota : (A', \mathcal{U}) \to (A, \mathcal{F})$ the inclusion morphism. Then $A' \in \mathcal{U}$ and, hence, $\alpha \circ \iota \neq \beta \circ \iota$. □

2.2. $\mathbb{U}$ on its own.

Definition 2.3. The *objects* in $\mathbb{U}$ are pairs $(A,\mathcal{U})$, where $\mathcal{U}$ is an ultrafilter on a nonempty set A. The *morphisms* $\alpha : (B,\mathcal{V}) \to (A,\mathcal{U})$ are equivalence classes of partial functions $\alpha : B \to A$ such that

(i) α is defined on some $V \in \mathcal{V}$,
(ii) $\alpha^{-1}(U) \in \mathcal{V}$, for all $U \in \mathcal{U}$ (α is continuous).

Two such partial functions $\alpha : V \to A$ and $\alpha' : V' \to A$ are equivalent if there is $E \subseteq V \cap V'$ such that $\alpha|_E = \alpha'|_E$.

The terminal object 1 in $\mathbb{U}$ is $(\{0\}, \{\{0\}\})$. The category $\mathbb{U}$ is not a very nice category, for instance it is neither closed under products nor pullbacks (see below). As a corollary of Proposition 1.6 we have the following characterization of being mono in $\mathbb{U}$:

Corollary 2.4. *Let $\alpha : \mathcal{V} \to \mathcal{U}$ be a morphism in $\mathbb{U}$. Then α is mono if and only if there is a $V \in \mathcal{V}$ such that α is injective on V.*

Proof. (i) "$\Longrightarrow$": Assume that $\alpha : \mathcal{V} \to \mathcal{U}$ is mono in $\mathbb{U}$ but not in $\mathbb{F}$. Then there are $\beta, \gamma : \mathcal{F} \to \mathcal{V}$ witnessing that α is not mono in $\mathbb{F}$,i.e. $\alpha \circ \beta = \alpha \circ \gamma$ but $\beta \neq \gamma$. Since $\mathbb{U}$ is generating $\mathbb{F}$, Proposition 2.2, there is an ultrafilter $\mathcal{W}$ and a morphism $\delta : \mathcal{W} \to \mathcal{F}$ such that $\beta \circ \delta \neq \gamma \circ \delta$. We still have $\alpha \circ (\beta \circ \delta) = \alpha \circ (\gamma \circ \delta)$ and, hence, α is not mono in $\mathbb{U}$. Contradiction. Since α is mono in $\mathbb{F}$ there is a $V \in \mathcal{V}$ such that α is injective on V.

"$\Longleftarrow$": If there is a $V \in \mathcal{V}$ such that α is injective on V then α is mono in $\mathbb{F}$. Assume that α is not mono in $\mathbb{U}$. Then there are an ultrafilter $\mathcal{W}$ and morphisms $\beta, \gamma : \mathcal{W} \to \mathcal{V}$ such that $\alpha \circ \beta = \alpha \circ \gamma$ but $\beta \neq \gamma$. But $\mathcal{W}$ is an object also in $\mathbb{F}$ and β and γ are morphisms also in $\mathbb{F}$ so α is not mono in $\mathbb{F}$ either. Hence, a contradiction. □

Proposition 2.5. *All morphisms $\alpha : (B,\mathcal{V}) \to (A,\mathcal{U})$ in $\mathbb{U}$ are epimorphisms.*

Proof. Show that for all $\alpha : (B,\mathcal{V}) \to (A,\mathcal{U})$ in $\mathbb{U}$ we have that $\alpha(\mathcal{V}) = \mathcal{U}$. Take $V \in \mathcal{V}$. If $A \setminus \alpha(V) \in \mathcal{U}$ then $\emptyset = \alpha^{-1}(A \setminus \alpha(V)) \cap V \in \mathcal{V}$. So $\alpha(V) \in \mathcal{U}$. For every $U \in \mathcal{U}$ we have $\alpha^{-1}(U) \in \mathcal{V}$ and $\alpha(\alpha^{-1}(U)) = U$. Thus $\alpha(\mathcal{V}) = \mathcal{U}$.

This means that all $\alpha : (B,\mathcal{V}) \to (A,\mathcal{U})$ are epi in $\mathbb{F}$ (by Proposition 1.6). Now, if there is a counter-example to α being epi in $\mathbb{U}$ then this counter-example holds true in $\mathbb{F}$ also and α is not epi in $\mathbb{F}$ either. □

Note that, in general, the product filter (in $\mathbb{F}$) of two ultrafilters is not an ultrafilter. For instance if $\mathcal{U}$ and $\mathcal{V}$ are two countably incomplete ultrafilters (i.e. not closed under countable intersections) then $\mathcal{U} \times \mathcal{V}$ is not an ultrafilter (see W.W. Comfort and S. Negrepontis [4, Cor. 7.24]). It could still be the case that $\mathbb{U}$ has products, but it is not.

Proposition 2.6. $\mathbb{U}$ *does not have products.*

Proof. Consider two ultrafilters $(A,\mathcal{U})$ and $(B,\mathcal{V})$ such that the product filter $(C, \mathcal{U} \times_{\mathbb{F}} \mathcal{V})$ is not an ultrafilter. Call the corresponding projections

$$\pi^1_{\mathbb{F}} : \mathcal{U} \times_{\mathbb{F}} \mathcal{V} \to \mathcal{U}$$

and

$$\pi^2_{\mathbb{F}} : \mathcal{U} \times_{\mathbb{F}} \mathcal{V} \to \mathcal{V}.$$

Let $U \subseteq C$ such that $U \notin \mathcal{U} \times_{\mathbb{F}} \mathcal{V}$ and $C \setminus U \notin \mathcal{U} \times_{\mathbb{F}} \mathcal{V}$.

Now extend the filter $\mathcal{U} \times_{\mathbb{F}} \mathcal{V}$ to two different ultrafilters $\mathcal{W}$ and $\mathcal{W}'$ such that $U \in \mathcal{W}$ and $C \setminus U \in \mathcal{W}'$. We have the morphisms $\iota : \mathcal{W} \to \mathcal{U} \times_{\mathbb{F}} \mathcal{V}$ and $\iota' : \mathcal{W}' \to \mathcal{U} \times_{\mathbb{F}} \mathcal{V}$, both given by the identity on C.

Assume that $\mathbb{U}$ has products. Let $(D, \mathcal{U} \times_{\mathbb{U}} \mathcal{V})$ be the product of $\mathcal{U}$ and $\mathcal{V}$ with projections

$$\pi^1_{\mathbb{U}} : \mathcal{U} \times_{\mathbb{U}} \mathcal{V} \to \mathcal{U}$$

and

$$\pi^2_{\mathbb{U}} : \mathcal{U} \times_{\mathbb{U}} \mathcal{V} \to \mathcal{V}.$$

$\mathcal{U} \times_{\mathbb{U}} \mathcal{V}$ is a filter and thus there is a unique morphism

$$\alpha : \mathcal{U} \times_{\mathbb{U}} \mathcal{V} \to \mathcal{U} \times_{\mathbb{F}} \mathcal{V}$$

such that $\pi^1_{\mathbb{F}} \circ \alpha = \pi^1_{\mathbb{U}}$ and $\pi^2_{\mathbb{F}} \circ \alpha = \pi^2_{\mathbb{U}}$.

The filters $\mathcal{W}$ and $\mathcal{W}'$ are ultrafilters and thus there are unique morphisms

$$\beta : \mathcal{W} \to \mathcal{U} \times_{\mathbb{U}} \mathcal{V}$$

such that $\pi^1_{\mathbb{U}} \circ \beta = \pi^1_{\mathbb{F}} \circ \iota$ and $\pi^2_{\mathbb{U}} \circ \beta = \pi^2_{\mathbb{F}} \circ \iota$ and

$$\gamma : \mathcal{W}' \to \mathcal{U} \times_{\mathbb{U}} \mathcal{V}$$

such that $\pi^1_{\mathbb{U}} \circ \gamma = \pi^1_{\mathbb{F}} \circ \iota'$ and $\pi^2_{\mathbb{U}} \circ \gamma = \pi^2_{\mathbb{F}} \circ \iota'$.

But $\iota^{(\prime)} : \mathcal{W}^{(\prime)} \to \mathcal{U} \times_{\mathbb{F}} \mathcal{V}$ are the unique morphisms $\delta^{(\prime)}$ such that $\pi^i_{\mathbb{F}} \circ \delta^{(\prime)} = \pi^i_{\mathbb{F}} \circ \iota^{(\prime)}$, for $i = 1, 2$. We have that $(\pi^1_{\mathbb{F}} \circ \alpha) \circ \beta = \pi^1_{\mathbb{U}} \circ \beta = \pi^1_{\mathbb{F}} \circ \iota$ and $(\pi^2_{\mathbb{F}} \circ \alpha) \circ \beta = \pi^2_{\mathbb{U}} \circ \beta = \pi^2_{\mathbb{F}} \circ \iota$ and, hence $\alpha \circ \beta = \iota$. In the same way, we get $\alpha \circ \gamma = \iota'$.

Now $\alpha^{-1}(U) \subseteq D$ and we have two cases:

(i) $\alpha^{-1}(U) \in \mathcal{U} \times_{\mathbb{U}} \mathcal{V}$. Then $\gamma^{-1}(\alpha^{-1}(U)) = U \in \mathcal{W}'$. A contradiction.

(ii) $D \setminus \alpha^{-1}(U) \in \mathcal{U} \times_{\mathbb{U}} \mathcal{V}$. Then $\beta^{-1}(D \setminus \alpha^{-1}(U)) = C \setminus U \in \mathcal{W}$. A contradiction.

Thus $\mathcal{U}$ and $\mathcal{V}$ has no product in $\mathbb{U}$. □

Since we have a terminal object in $\mathbb{U}$ this also means that there are in general no pullbacks either.

Now follows some facts about ultrafilters and the consequences they have for the objects in the category $\mathbb{U}$. For more information see for instance Comfort and Negrepontis [4].

Definition 2.7. The *norm* of an ultrafilter $\mathcal{U}$, denoted by $||\mathcal{U}||$, is defined by

$$||\mathcal{U}|| = \min\{|U| : U \in \mathcal{U}\}.$$

Definition 2.8. If $(A, \mathcal{U})$ is an ultrafilter then $\mathcal{U}$ is *uniform* if $||\mathcal{U}|| = |A|$.

Note that for uniform ultrafilters we have: $\forall U \in \mathcal{U}\ |U| = |A|$. As an example any ultrafilter extending the co-finite filter on $\mathbb{N}$ is uniform.

Proposition 2.9. *For every set A there exists a uniform ultrafilter on A.*

Proposition 2.10. *Every ultrafilter $(A, \mathcal{U})$ is isomorphic to a uniform ultrafilter.*

Proof. Take $U \in \mathcal{U}$ such that $|U| = ||\mathcal{U}||$. The inclusion $i : (U, \mathcal{U} \restriction U) \to (A, \mathcal{U})$ induces an isomorphism and $(U, \mathcal{U} \restriction U)$ is a uniform ultrafilter. □

This observation can be found in Koubek and Reiterman [7]:

Proposition 2.11. *If* $\mathcal{U}$ *and* $\mathcal{V}$ *are uniform ultrafilters and* $||\mathcal{U}|| < ||\mathcal{V}||$ *then* $\mathrm{Hom}_{\mathbb{U}}(\mathcal{U}, \mathcal{V}) = \emptyset$.

Proof. Assume that there is a morphism $\alpha : \mathcal{U} \to \mathcal{V}$ and assume that α is defined on $U \in \mathcal{U}$. Then $\alpha(U) \in \mathcal{V}$ and, thus $|\alpha(U)| = ||\mathcal{V}||$. But we have $|U| = ||\mathcal{U}|| < ||\mathcal{V}||$. Contradiction. □

2.3. **The Rudin-Keisler ordering.** The structure, in a broad sense, of $\mathbb{U}$ has been studied in a different guise, namely as ultrafilters ordered by the Rudin-Keisler ordering. The following definition can be found in C.C. Chang and H.J. Keisler [3, Ex. 4.3.40].

Definition 2.12. The *Rudin-Keisler ordering*: If $(A, \mathcal{U})$ and $(B, \mathcal{V})$ are ultrafilters then

$$\mathcal{U} \leq \mathcal{V} \iff \exists f : B \to A \text{ such that } \mathcal{U} = \{X \subseteq A \mid f^{-1}(X) \in \mathcal{V}\}.$$

We will write $f^{-1}[\mathcal{U}] = \mathcal{V}$ for the righthandside condition.

The proof of the following proposition can be found in Comfort and Negrepontis [4, Chapter 9].

Proposition 2.13. *If* $(A, \mathcal{U})$ *is an ultrafilter and* $f : A \to A$ *a function then*

$$f^{-1}[\mathcal{U}] = \mathcal{U} \iff \{x \in A \mid f(x) = x\} \in \mathcal{U}.$$

Corollary 2.14. *If, for ultrafilters* $(A, \mathcal{U})$ *and* $(B, \mathcal{V})$ *and functions* $f : B \to A$ *and* $g : A \to B$ *we have* $f^{-1}[\mathcal{U}] = \mathcal{V}$ *and* $g^{-1}[\mathcal{V}] = \mathcal{U}$*, then there is a* $V \in \mathcal{V}$ *such that* f *is one-to-one on* V*.*

Proof. We have $g \circ f : B \to B$ and $(g \circ f)^{-1}[\mathcal{V}] = \mathcal{V}$ and hence, by the Proposition, $V = \{x \in B \mid g(f(x)) = x\} \in \mathcal{V}$. Assume that we have $x_0, x_1 \in V$ such that $f(x_0) = f(x_1)$. Then we have $g(f(x_0)) = g(f(x_1))$ and hence $x_0 = x_1$. □

Proposition 2.15. $\mathcal{U} \leq \mathcal{V} \iff$ *there is a morphism* $\alpha : (B, \mathcal{V}) \to (A, \mathcal{U})$ *in* $\mathbb{U}$.

Proof. "$\Longrightarrow$": Take $f : B \to A$ such that $\mathcal{U} = \{X \subseteq A \mid f^{-1}(X) \in \mathcal{V}\}$. Define α as $[f]$ and show that it is continuous. Let $\alpha : V \to A$, for some $V \in \mathcal{V}$. Take $U \in \mathcal{U}$. Then $f^{-1}(U) \in \mathcal{V}$, and hence $\alpha^{-1}(U) = f^{-1}(U) \cap V \in \mathcal{V}$.

"$\Longleftarrow$": Let $\alpha : V \to A$, for some $V \in \mathcal{V}$. Define $f : B \to A$ as any function such that $f|_V = \alpha$. If $U \in \mathcal{U}$ then $\alpha^{-1}(U) \in \mathcal{V}$, since α continuous, and $\alpha^{-1}(U) \subseteq f^{-1}(U)$ implies $f^{-1}(U) \in \mathcal{V}$. If $X \subseteq A$ and $f^{-1}(X) \in \mathcal{V}$ then $\alpha(f^{-1}(X)) \in \mathcal{U}$ and $\alpha(f^{-1}(X)) \subseteq f(f^{-1}(X))$ implies $X = f(f^{-1}(X)) \in \mathcal{U}$. □

Proposition 2.16. $\mathcal{U} \leq \mathcal{V}$ *and* $\mathcal{V} \leq \mathcal{U} \iff (A, \mathcal{U})$ *and* $(B, \mathcal{V})$ *are isomorphic in* $\mathbb{U}$.

Proof. "$\Longrightarrow$": Assume $f : B \to A$ is such that $f^{-1}[\mathcal{U}] = \mathcal{V}$. Then, by Corollary 2.14, f is one-to-one on some $V \in \mathcal{V}$. Then $[f] : (B, \mathcal{V}) \to (A, \mathcal{U})$ is a monomorphism, and thus an isomorphism.

"$\Longleftarrow$": Clear from Proposition 2.15. □

The Rudin-Keisler ordering is a partial ordering on the isomorphism classes of $\mathbb{U}$. We gather some relevant results on the Rudin-Keisler ordering in this theorem. These will be true assuming, for instance, the generalized continuum hypothesis (GCH) (for proofs see Comfort and Negrepontis [4, Chapter 9]).

Theorem 2.17. *Let A be a set, with $|A| = \kappa$, and consider the Rudin-Keisler ordering restricted to A. Then*

(i) *There are 2^{2^κ} non-isomorphic minimal ultrafilters on A.*
(ii) *There are no maximal ultrafilters on A.*

This gives that there are 2^{2^ω} minimal ultrafilters above 1 and no maximal ultrafilters in the Rudin-Keisler ordering on $\mathbb{U}$.

3. The sheaves on the category of ultrafilters

3.1. **The Grothendieck topology on $\mathbb{U}$.** We define a Grothendieck topology on $\mathbb{U}$. This topology will be the same as the topology induced on $\mathbb{U}$ from $\mathbb{F}$.

Definition 3.1. A morphism $\alpha : \mathcal{V} \to \mathcal{U}$ is a *covering* if $\alpha(V) \in \mathcal{U}$, for all $V \in \mathcal{V}$.

Note that $\alpha : \mathcal{V} \to \mathcal{U}$ is a covering if and only if it is an epimorphism. Thus, by Proposition 2.5, all morphisms in $\mathbb{U}$ are coverings. By the following lemma these coverings constitute a basis for a Grothendieck topology, the *atomic topology*, on $\mathbb{U}$:

Lemma 3.2. (i) *If $\alpha : \mathcal{V} \to \mathcal{U}$ is an isomorphism then $\alpha : \mathcal{V} \to \mathcal{U}$ is a covering.*
(ii) *If $\alpha : \mathcal{V} \to \mathcal{U}$ is a covering then, for every $\beta : \mathcal{W} \to \mathcal{U}$ there is a covering $\gamma : \mathcal{W}' \to \mathcal{W}$ such that $\beta \circ \gamma$ factorizes through α.*
(iii) *If $\alpha : \mathcal{V} \to \mathcal{U}$ and $\beta : \mathcal{W} \to \mathcal{V}$ are coverings then $\alpha \circ \beta : \mathcal{W} \to \mathcal{U}$ is a covering.*

Proof. (i) Clear from the definition of covering.

(ii) Let $\widehat{\mathcal{W}}$ be the pullback of $\alpha : \mathcal{V} \to \mathcal{U}$ and $\beta : \mathcal{W} \to \mathcal{U}$ in the category $\mathbb{F}$. Then there are projections $\pi_1 : \widehat{\mathcal{W}} \to \mathcal{W}$ and $\pi_2 : \widehat{\mathcal{W}} \to \mathcal{V}$ such that $\beta \circ \pi_1 = \alpha \circ \pi_2$. Since $\alpha : \mathcal{V} \to \mathcal{U}$ is a covering (in $\mathbb{F}$) we have that $\pi_2 : \widehat{\mathcal{W}} \to \mathcal{V}$ is a covering and, hence $\widehat{\mathcal{W}}$ is a proper filter. Now expand $\widehat{\mathcal{W}}$ to an ultrafilter $\mathcal{W}'$. Then $\pi_1 : \mathcal{W}' \to \mathcal{W}$ is the sought after covering and $\beta \circ \pi_1$ factors through α.

(iii) Clear from the definition of covering. □

The category $\mathbb{U}$ with the Grothendieck topology J, generated by the coverings, constitute a site, and give rise to the Grothendieck topos $\mathrm{Sh}(\mathbb{U}, J)$. Since the topology J is atomic then, for every sheaf E on $\mathbb{U}$, the lattice of all subsheaves of E is an atomic complete Boolean algebra. We will also later make use of the fact that there are no empty coverings in J (for both facts, see Mac Lane and Moerdijk [8, section III.8]).

3.2. **Sheaves on $\mathbb{U}$.** Let us investigate which sheaves we have on $\mathbb{U}$.

First we will just state the conditions on a presheaf to be a sheaf in the topology on $\mathbb{U}$:

Lemma 3.3. *Let P be a presheaf on $\mathbb{U}$ (i.e. a contravariant functor from $\mathbb{U}$ to* **Sets***).*

Then P is a sheaf if and only if for every $\alpha : \mathcal{V} \to \mathcal{U}$ in $\mathbb{U}$ and every $y \in P(\mathcal{V})$, if $\alpha \circ \beta = \alpha \circ \gamma$ (with $\beta, \gamma : \mathcal{W} \to \mathcal{V}$) implies that $P(\beta)(y) = P(\gamma)(y)$, then there is a unique $x \in P(\mathcal{U})$ such that $P(\alpha)(x) = y$.

If we already have a sheaf S then finding its subsheaves is somewhat easier.

Lemma 3.4. *Let S be a sheaf and P a subpresheaf of S (i.e. $P(\mathcal{U}) \subseteq S(\mathcal{U})$ and $P(\alpha) = S(\alpha)|_{P(\mathcal{U})}$, for all $\mathcal{U}$ and $\alpha : \mathcal{V} \to \mathcal{U}$ in $\mathbb{U}$).*

Then P is a sheaf if and only if for every $\mathcal{U} \in \mathbb{U}$, $x \in S(\mathcal{U})$ and $\alpha : \mathcal{V} \to \mathcal{U}$, $S(\alpha)(x) \in P(\mathcal{V})$ implies that $x \in P(\mathcal{U})$.

Theorem 3.5. *Let $P = \mathrm{Hom}_{\mathbb{F}}(-, \mathcal{F}) : \mathbb{F}^{op} \to$ **Sets** be a sheaf on $\mathbb{F}$. Then the restriction of P to $\mathbb{U}$ is a sheaf on $\mathbb{U}$.*

Proof. Assume that $\alpha : \mathcal{V} \to \mathcal{U}$ is a morphism in $\mathbb{U}$. Take $y \in P(\mathcal{V})$ such that $P(\beta)(y) = P(\gamma)(y)$, for all $\beta, \gamma : \mathcal{W} \to \mathcal{V}$ with $\alpha \circ \beta = \alpha \circ \gamma$ in $\mathbb{U}$. For every such pair β, γ there is an unique $\tau : \mathcal{W} \to \mathcal{V} \times_{\mathcal{U}} \mathcal{V}$ in $\mathbb{F}$ such that $\beta = \pi_1 \circ \tau$ and $\gamma = \pi_2 \circ \tau$.

We have to show that this $y \in P(\mathcal{V})$ is a matching family for the covering $\alpha : \mathcal{V} \to \mathcal{U}$ in $\mathbb{F}$. In order to do this, show that $P(\pi_1)(y) = P(\pi_2)(y)$, i.e. $y \circ \pi_1 = y \circ \pi_2$. Take an ultrafilter $\mathcal{W}$ and $\sigma : \mathcal{W} \to \mathcal{V} \times_{\mathcal{U}} \mathcal{V}$. Let $\beta = \pi_1 \circ \sigma$ and $\gamma = \pi_2 \circ \sigma$. Then $\alpha \circ \beta = \alpha \circ \gamma$ since $\mathcal{V} \times_{\mathcal{U}} \mathcal{V}$ is a pullback and thus, by assumption $y \circ \beta = y \circ \gamma$. This means that $y \circ \pi_1 \circ \sigma = y \circ \pi_2 \circ \sigma$ and then, by Proposition 2.2, $y \circ \pi_1 = y \circ \pi_2$, since $\mathcal{W}$ and σ were arbitrary. Since P is a sheaf on $\mathbb{F}$ there is an unique $x \in P(\mathcal{U})$ s.t. $y = P(\alpha)(x)$. This shows that $P : \mathbb{U} \to$ **Sets** is a sheaf. □

For every set S there is a representable sheaf ${}^*S = \mathrm{Hom}_{\mathbb{F}}(-, (S, \{S\}))$ on $\mathbb{F}$. Recall that if you have a morphism $\alpha : \mathcal{V} \to \mathcal{U}$ then ${}^*S(\alpha) : {}^*S(\mathcal{U}) \to {}^*S(\mathcal{V})$ is defined as ${}^*S(\alpha)(\beta) = \beta \circ \alpha : \mathcal{V} \to S$, for $\beta : \mathcal{U} \to S$.

Corollary 3.6. *The ${}^*S$ are sheaves on $\mathbb{U}$.*

Corollary 3.7. *The topology J on $\mathbb{U}$ is subcanonical.*

Proof. The representable presheaves on $\mathbb{U}$ are on the form $\mathrm{Hom}_{\mathbb{U}}(-, \mathcal{U})$, where the Hom-sets coincide with the Hom-sets over $\mathbb{F}$. □

Since the topology on $\mathbb{U}$ is subcanonical the Yoneda embedding goes into $\mathrm{Sh}(\mathbb{U})$ and is a full and faithful functor, $\mathbf{y} : \mathbb{U} \to \mathrm{Sh}(\mathbb{U})$, defined at $\mathcal{U}$ by $\mathbf{y}(\mathbb{U}) = \mathrm{Hom}_{\mathcal{U}}(\,\cdot\,, \mathcal{U})$. With the help of this functor we can define the Rudin-Keisler ordering on the representable sheaves on $\mathbb{U}$. First, we define a preorder on $\mathrm{Sh}(\mathbb{U})$:

Definition 3.8. For any X and Y in $\mathrm{Sh}(\mathbb{U})$ we define

$$X \leq Y \Longleftrightarrow \mathrm{Hom}_{\mathrm{Sh}(\mathbb{U})}(Y, X) \neq \emptyset.$$

From the following lemma, we conclude that this order is a partial order on the representable sheaves on $\mathbb{U}$.

Lemma 3.9. *For any representable sheaves $\mathbf{y}(\mathcal{U})$ and $\mathbf{y}(\mathcal{V})$ in $\mathrm{Sh}(\mathbb{U})$ (i.e. $\mathcal{U}, \mathcal{V} \in \mathbb{U}$) we have*

$$\mathbf{y}(\mathcal{U}) \leq \mathbf{y}(\mathcal{V}) \Longleftrightarrow \mathcal{U} \leq_{RK} \mathcal{V},$$

where $\leq_{RK}$ is the Rudin-Keisler ordering on $\mathbb{U}$.

Proof. Since $\mathbf{y} : \mathbb{U} \to \mathrm{Sh}(\mathbb{U})$ is full and faithful we have a bijection between $\mathrm{Hom}_{\mathrm{Sh}(\mathbb{U})}(\mathbf{y}(\mathcal{U}), \mathbf{y}(\mathcal{V}))$ and $\mathrm{Hom}_{\mathcal{U}}(\mathcal{U}, \mathcal{V})$. □

Proposition 3.10. *For any $\mathcal{U}$ and $\mathcal{V}$ we have that $\mathbf{y}(\mathcal{U}) \leq \mathbf{y}(\mathcal{V})$ if and only if for any $\mathcal{W}$ we have $\mathbf{y}(\mathcal{V})(\mathcal{W}) \neq \emptyset$ implies $\mathbf{y}(\mathcal{U})(\mathcal{W}) \neq \emptyset$.*

Proof. "$\Longrightarrow$": $\mathbf{y}(\mathcal{U}) \leq \mathbf{y}(\mathcal{V})$ means from the lemma above that there is an $\alpha : \mathcal{V} \to \mathcal{U}$. Take $\mathcal{W}$ such that $\mathbf{y}(\mathcal{V})(\mathcal{W}) \neq \emptyset$, i.e. there is a $\beta : \mathcal{W} \to \mathcal{V}$. Now $\alpha \circ \beta : \mathcal{W} \to \mathcal{U} \in \mathbf{y}(\mathcal{U})(\mathcal{W})$.

"$\Longleftarrow$": We have that $\mathbf{y}(\mathcal{V})(\mathcal{V}) \neq \emptyset$. By assumption $\mathbf{y}(\mathcal{U})(\mathcal{V}) \neq \emptyset$, and hence $\mathcal{U} \leq \mathcal{V}$. □

The terminal object in $\mathrm{Sh}(\mathbb{U})$ is the sheaf 1, $1(\mathcal{U}) = \{0\}$ and $1(\alpha) = id : \{0\} \to \{0\}$ for all $\mathcal{U}$ and $\alpha : \mathcal{V} \to \mathcal{U}$ in $\mathbb{U}$. The initial object 0 is the empty functor, i.e. $0(\mathcal{U}) = \emptyset$ and $0(\alpha) = \emptyset : \emptyset \to \emptyset$ for all $\mathcal{U}$ and $\alpha : \mathcal{V} \to \mathcal{U}$ in $\mathbb{U}$. This is true since the topology on $\mathbb{U}$ is atomic, and, hence, contains no empty coverings. We prove that the lattice of subsheaves of the sheaf 1 has a particular simple form, namely that of a two-point set.

Theorem 3.11. *The topos* $\mathrm{Sh}(\mathbb{U})$ *is two-valued, i.e. the sheaves* 1 *and* 0 *are the only subsheaves of* 1*.*

Proof. Assume that S is a subsheaf of 1 and that there is a $\mathcal{U}_0 \in \mathbb{U}$ such that $S(\mathcal{U}_0) \neq \emptyset$. Since S is a subsheaf of 1 this means $S(\mathcal{U}_0) = \{0\}$. Then if there is a morphism $\alpha : \mathcal{V} \to \mathcal{U}_0$ then $S(\mathcal{V}) \neq \emptyset$ since $S(\alpha) : S(\mathcal{U}_0) \to S(\mathcal{V})$ is a function.

If there is a morphism $\beta : \mathcal{U}_0 \to \mathcal{V}$ then consider $0 \in 1(\mathcal{V})$. We have that $1(\beta)(0) = 0 \in S(\mathcal{U}_0)$ and hence, by Lemma 3.4, $0 \in S(\mathcal{V})$. Since all ultrafilters in $\mathbb{U}$ are connected by morphisms this proves that $S = 1$. □

4. The logic in $\mathrm{Sh}(\mathbb{U})$

4.1. **The sheaf semantics.** Assume that L is a first order language including symbols for all sets, relations, functions and constants of interest for us. Then *L denotes the language where all symbols have been decorated by *.

These new symbols will be interpreted as follows:

For each set S the symbol $^*S$ will be interpreted as the sheaf $^*S$ (see Corollary 3.6). This sheaf will be called the *nonstandard version* of S.

For each relation $R \subseteq S_1 \times \ldots \times S_n$ the symbol $^*R$ will be interpreted as the subsheaf of $^*S_1 \times \ldots \times {}^*S_n$, given at $\mathcal{U}$ by the definition

$$(\alpha_1, \ldots, \alpha_n) \in {}^*R(\mathcal{U}) \Longleftrightarrow (\exists U \in \mathcal{U})(\forall x \in U)(\alpha_1(x), \ldots, \alpha_n(x)) \in R.$$

If $f : S_1 \times \ldots \times S_n \to S$ then the symbol $^*f$ will be interpreted at $\mathcal{U}$ as the internal function from $^*S_1(\mathcal{U}) \times \ldots \times {}^*S_n(\mathcal{U})$ to $^*S(\mathcal{U})$ given by

$$^*f_{\mathcal{U}}(\alpha_1, \ldots, \alpha_n) = \lambda x. f(\alpha_1(x), \ldots, \alpha_n(x)).$$

For each constant $s \in S$ the symbol $^*s$ will be interpreted as the constant function $\lambda x.s \in {}^*S(\mathcal{U})$. The $^*s$, for $s \in S$, are called the *standard* elements of $^*S$. The sheaf of standard elements of $^*S$ is denoted $\Delta(S)$.

Note that this definition gives that $^*S(\mathcal{U})$ can be considered not just as a set but as the ultrapower of a structure on S over $\mathcal{U}$.

For any L-formula Θ, we define its *-transform, $^*\Theta$, to be the *L-formula where all symbols have been replaced by their starred counterparts. A formula which is the *-transform of a L-formula is called *internal*. All other formulas will be called *external*. The language *L can be regarded as a sublanguage of the language $\mathrm{L}(\mathrm{Sh}(\mathbb{U}))$ of the topos $\mathrm{Sh}(\mathbb{U})$.

Since we want to be able to quantify over the category $\mathbb{U}$, we have to make it into a set. Formally, this is done by introducing an universe of sets into set theory, e.g. V_κ, where κ is an inaccessible cardinal. The sets belonging to the universe are called *small*.

As mentioned above, in Section 3.1, the lattice of all subsheaves of a sheaf E is an Boolean algebra. This means, stated differently, that the internal logic in $\mathrm{Sh}(\mathbb{U})$ is classic.

Theorem 4.1. *The topos* $\mathrm{Sh}(\mathbb{U})$ *is Boolean, i.e. for every* $\mathcal{U} \in \mathbb{U}$ *we have*

$$\mathcal{U} \Vdash (\forall x)(\Theta(x) \vee \neg\Theta(x)),$$

for all formulas $\Theta(x)$.

In the topos $\mathrm{Sh}(\mathbb{U})$ the sheaf semantics have the following form:

Theorem 4.2. *Let* $\mathcal{U}$ *be an ultrafilter, and let* Φ, Ψ *be arbitrary external formulas. We write* $\overline{\alpha} = \alpha_1, \ldots, \alpha_n$*, where* $\alpha_1 \in {}^*S_1(\mathcal{U}), \ldots, \alpha_n \in {}^*S_n(\mathcal{U})$*. Then*

(i) $\mathcal{U} \Vdash \Phi(\overline{\alpha}) \wedge \Psi(\overline{\alpha})$ *if and only if* $\mathcal{U} \Vdash \Phi(\overline{\alpha})$ *and* $\mathcal{U} \Vdash \Psi(\overline{\alpha})$*,*

(ii) $\mathcal{U} \Vdash \Phi(\overline{\alpha}) \vee \Psi(\overline{\alpha})$ *if and only if for some* $\beta : \mathcal{V} \to \mathcal{U}$ $\mathcal{V} \Vdash \Phi(\overline{\alpha}\circ\beta)$ *or* $\mathcal{V} \Vdash \Psi(\overline{\alpha}\circ\beta)$*,*

(iii) $\mathcal{U} \Vdash \Phi(\overline{\alpha}) \to \Psi(\overline{\alpha})$ *if and only if for all* $\beta : \mathcal{V} \to \mathcal{U}$ $\mathcal{V} \Vdash \Phi(\overline{\alpha} \circ \beta)$ *implies* $\mathcal{V} \Vdash \Psi(\overline{\alpha} \circ \beta)$*,*

(iv) $\mathcal{U} \Vdash \neg\Phi(\overline{\alpha})$ *if and only if* $\mathcal{U} \nVdash \Phi(\overline{\alpha})$*,*

(v) $\mathcal{U} \Vdash (\exists x \in {}^*S)\Phi(\overline{\alpha}, x)$ *if and only if for some* $\beta : \mathcal{V} \to \mathcal{U}$ *and* $\delta \in {}^*S(\mathcal{V})$ $\mathcal{V} \Vdash \Phi(\overline{\alpha} \circ \beta, \delta)$*,*

(vi) $\mathcal{U} \Vdash (\forall x \in {}^*S)\Phi(\overline{\alpha}, y)$ *if and only if for all* $\beta : \mathcal{V} \to \mathcal{U}$ *and* $\delta \in {}^*S(\mathcal{V})$ $\mathcal{V} \Vdash \Phi(\overline{\alpha} \circ \beta, \delta)$*.*

Here in (iv) we use the fact that there are no empty coverings in the atomic topology.
We also define what it means to be standard for a $\gamma \in {}^*S(\mathcal{U})$:
(vii) $\mathcal{U} \Vdash \mathrm{St}(\gamma)$ *if and only if* γ *is constant on some* $U \in \mathcal{U}$*.*
To be precise one should use the notation "$\mathrm{St}^{{}^*S}(\gamma)$" but we will usually omit the superscript. We will write ${}^*\mathrm{L}+\mathrm{St}$ for the language ${}^*\mathrm{L}$ with the predicates $\mathrm{St}^{{}^*S}$ added, for all ${}^*S$ in ${}^*\mathrm{L}$. We will use the usual abbreviations $(\forall^{st} x \in {}^*S)\ldots$ for $(\forall x \in {}^*S)\,\mathrm{St}(x) \to \ldots$ and $(\exists^{st} x \in {}^*S)\ldots$ for $(\exists x \in {}^*S)\,\mathrm{St}(x) \wedge \ldots$. Note that we have the following characterization for a $\gamma \in {}^*S(\mathcal{U})$:

$$\mathcal{U} \Vdash \mathrm{St}(\gamma) \text{ if and only if } \gamma = {}^*s, \text{ for some } s \in S.$$

The forcing relation has these two additional properties:

(i) (Monotonicity) If $\mathcal{U} \Vdash \Phi(\overline{\alpha})$ and $\beta : \mathcal{V} \to \mathcal{U}$ then $\mathcal{V} \Vdash \Phi(\overline{\alpha} \circ \beta)$.

(ii) (Local character) If $\beta : \mathcal{V} \to \mathcal{U}$ and $\mathcal{V} \Vdash \Phi(\overline{\alpha} \circ \beta)$ then $\mathcal{U} \Vdash \Phi(\overline{\alpha})$.

We can simplify some of the clauses in the above theorem:

Theorem 4.3. *Let* $\mathcal{U}$ *be an ultrafilter, and let* Φ, Ψ *be arbitrary external formulas. Then*

(i) $\mathcal{U} \Vdash \Phi(\overline{\alpha}) \vee \Psi(\overline{\alpha})$ *if and only if* $\mathcal{U} \Vdash \Phi(\overline{\alpha})$ *or* $\mathcal{U} \Vdash \Psi(\overline{\alpha})$*,*

(ii) $\mathcal{U} \Vdash \Phi(\overline{\alpha}) \to \Psi(\overline{\alpha})$ *if and only if* $\mathcal{U} \Vdash \Phi(\overline{\alpha})$ *implies* $\mathcal{U} \Vdash \Psi(\overline{\alpha})$*,*

(iii) $\mathcal{U} \Vdash (\exists^{st} y \in {}^*S)\Phi(\overline{\alpha}, y)$ *if and only if for some* $s \in S$ $\mathcal{U} \Vdash \Phi(\overline{\alpha}, {}^*s)$*,*

(iv) $\mathcal{U} \Vdash (\forall^{st} y \in {}^*S)\Phi(\overline{\alpha}, y)$ *if and only if for all* $s \in S$ $\mathcal{U} \Vdash \Phi(\overline{\alpha}, {}^*s)$*.*

Proof. (i) "$\Longrightarrow$" $\mathcal{U} \Vdash \Phi(\overline{\alpha}) \vee \Psi(\overline{\alpha})$ implies that there is $\beta : \mathcal{V} \to \mathcal{U}$ such that $\mathcal{V} \Vdash \Phi(\overline{\alpha}\circ\beta)$ or $\mathcal{V} \Vdash \Psi(\overline{\alpha} \circ \beta)$. By local character we have that $\mathcal{U} \Vdash \Phi(\overline{\alpha})$ or $\mathcal{U} \Vdash \Psi(\overline{\alpha})$.

(ii) "$\Longleftarrow$" Take $\beta : \mathcal{V} \to \mathcal{U}$. Assume that $\mathcal{V} \Vdash \Phi(\overline{\alpha} \circ \beta)$. Local character gives that $\mathcal{U} \Vdash \Phi(\overline{\alpha})$. Then we have $\mathcal{U} \Vdash \Psi(\overline{\alpha})$ and by monotonicity $\mathcal{V} \Vdash \Psi(\overline{\alpha} \circ \beta)$.

(iii) "$\Longrightarrow$" $\mathcal{U} \Vdash (\exists^{st} y \in {}^*S)\Phi(\overline{\alpha}, y)$ gives that there is $\beta : \mathcal{V} \to \mathcal{U}$ and $\delta \in {}^*S(\mathcal{V})$ s.t. $\mathcal{V} \Vdash \mathrm{St}(\delta) \wedge \Phi(\overline{\alpha} \circ \beta, \delta)$. That is: $\mathcal{V} \Vdash \Phi(\overline{\alpha}, {}^*s)$, for some $s \in S$. By local character we have, for some $s \in S$, $\mathcal{U} \Vdash \Phi(\overline{\alpha}, {}^*s)$.

(iv) "$\Longleftarrow$" Assume $\mathcal{U} \Vdash \Phi(\overline{\alpha}, {}^*s)$, for all $s \in S$. Take any $\beta : \mathcal{V} \to \mathcal{U}$ and $\delta \in {}^*S(\mathcal{V})$. If $\mathcal{V} \Vdash \mathrm{St}(\delta)$ then $\delta = {}^*s$, for some $s \in S$. Hence we get, by monotonicity, $\mathcal{V} \Vdash \mathrm{St}(\delta) \to \Phi(\overline{\alpha} \circ \beta, \delta)$. □

With the help of the additional structure given by the language *L we also have an alternative characterization of the Rudin-Keisler ordering on $\mathbb{U}$. With a structure ${}^*\mathbf{S}$ at an ultrafilter $\mathcal{U}$ we mean a sheaf ${}^*S$ together with the language *L restricted to ${}^*S(\mathcal{U})$.

An example: if ${}^*L = ({}^*S, {}^*\mathbb{R}, {}^*+, {}^*>, {}^*0)$ then a structure ${}^*\mathbf{S}$ at some $\mathcal{U}$ is $({}^*S(\mathcal{U}); {}^*\mathbb{R}(\mathcal{U}), {}^*+, {}^*>, {}^*0)$, where ${}^*+$ and ${}^*>$ are restricted to ${}^*S(\mathcal{U})$. Notice that all formulas in ${}^*\mathbf{S}$ are internal.

Proposition 4.4. *$\mathcal{U} \leq \mathcal{V}$ if and only if for any structure ${}^*\mathbf{S}$ there is an elementary embedding ${}^*\mathbf{S}(\mathcal{U}) \prec {}^*\mathbf{S}(\mathcal{V})$.*

The proof of this proposition can be found in Comfort and Negrepontis [4, Thm. 13.24]. We have that there is an elementary embedding ${}^*\mathbf{S}(\mathcal{U}) \prec {}^*\mathbf{S}(\mathcal{V})$ if and only if there exists a function $f : {}^*S(\mathcal{U}) \to {}^*S(\mathcal{V})$ such that for any formula ${}^*\Theta(\overline{x})$ in ${}^*\mathbf{S}$ and elements $\overline{\alpha} \in {}^*S(\mathcal{U})$ we have

$$ {}^*\mathbf{S}(\mathcal{U}) \models {}^*\Theta(\overline{\alpha}) \Longleftrightarrow {}^*\mathbf{S}(\mathcal{V}) \models {}^*\Theta(f(\overline{\alpha})). $$

Lemma 4.5. *For any structure ${}^*\mathbf{S}$, ultrafilter $\mathcal{U}$ and formula ${}^*\Theta(\overline{x})$ in ${}^*\mathbf{S}$, we have*

$$ {}^*\mathbf{S}(\mathcal{U}) \models {}^*\Theta(\overline{\alpha}) \Longleftrightarrow \mathcal{U} \Vdash {}^*\Theta(\overline{\alpha}). $$

From this lemma follows a proof of the implication from left to right in the proposition: if $\mathcal{U} \leq \mathcal{V}$ then there is a morphism $\beta : \mathcal{V} \to \mathcal{U}$. By monotonicity follows that if $\mathcal{U} \Vdash {}^*\Theta(\overline{\alpha})$ then $\mathcal{V} \Vdash {}^*\Theta({}^*S(\beta)(\overline{\alpha}))$, since ${}^*S(\beta)(\overline{\alpha}) = \overline{\alpha} \circ \beta$. By local character we get that if $\mathcal{V} \Vdash {}^*\Theta({}^*S(\beta)(\overline{\alpha}))$ then $\mathcal{U} \Vdash {}^*\Theta(\overline{\alpha})$. Hence, by the lemma above ${}^*\mathbf{S}(\mathcal{U}) \prec {}^*\mathbf{S}(\mathcal{V})$.

In order to prove the lemma we need an additional result and the proof will come in the next section (Corollary 4.8). That the equivalence in the lemma is not true in general will be shown in section 4.5.

4.2. **Transfer principles.** An important theorem for ultraproducts is Łoś's theorem. The corresponding version in this setting is called Moerdijk's theorem and it holds for $\mathrm{Sh}(\mathbb{U})$:

Theorem 4.6. *Let $\mathcal{U}$ be an ultrafilter and Θ an L-formula. Then*

$$ \mathcal{U} \Vdash {}^*\Theta(\overline{\alpha}) \text{ if and only if } (\exists U \in \mathcal{U})(\forall x \in U)\Theta(\overline{\alpha}(x)). $$

Proof. Proof by induction on the formula ${}^*\Theta$.

- ${}^*\Theta(\overline{\alpha}) \equiv {}^*\Phi(\overline{\alpha}) \wedge {}^*\Psi(\overline{\alpha})$ Clear.
- ${}^*\Theta(\overline{\alpha}) \equiv \neg{}^*\Phi(\overline{\alpha})$

  "$\Longrightarrow$" Assume $\mathcal{U} \Vdash \neg{}^*\Phi(\overline{\alpha})$. Then $\mathcal{U} \not\Vdash {}^*\Phi(\overline{\alpha})$. By the induction hypothesis (IH): $(\forall U \in \mathcal{U})(\exists x \in U)\neg\Phi(\overline{\alpha}(x))$. Consider the set $U_1 = \{x \,|\, \neg\Phi(\overline{\alpha}(x))\}$. If

$U_1 \notin \mathcal{U}$ then $\{x \mid \Phi(\overline{\alpha}(x))\} \in \mathcal{U}$ but this contradicts the assumption above. Hence $(\exists U \in \mathcal{U})(\forall x \in U)\neg\Phi(\overline{\alpha}(x))$.

"$\Longleftarrow$" Assume $(\exists U_1 \in \mathcal{U})(\forall x \in U)\neg\Phi(\overline{\alpha}(x))$. If $\mathcal{U} \Vdash {}^*\Phi(\overline{\alpha})$ then, by IH, $(\exists U_2 \in \mathcal{U})(\forall x \in U)\Phi(\overline{\alpha}(x))$. But $\emptyset \neq U_1 \cap U_2 \in \mathcal{U}$ and, thus, a contradiction. Hence $\mathcal{U} \Vdash \neg{}^*\Phi(\overline{\alpha})$.

- ${}^*\Theta(\overline{\alpha}) \equiv (\exists x \in {}^*S){}^*\Phi(\overline{\alpha}, x)$

"$\Longrightarrow$" Assume $\mathcal{U} \Vdash (\exists x \in {}^*S){}^*\Phi(\overline{\alpha}, x)$. Then there is $\beta : \mathcal{V} \to \mathcal{U}$ and $\delta \in {}^*S(\mathcal{V})$ such that $\mathcal{V} \Vdash {}^*\Phi(\overline{\alpha} \circ \beta, \delta)$. Now by IH: $(\exists V \in \mathcal{V})(\forall x \in V)\Phi(\overline{\alpha}(\beta(x)), \delta(x))$. $\beta : \mathcal{V} \to \mathcal{U}$ is a covering so if $V \in \mathcal{V}$ then $\beta(V) \in \mathcal{U}$. But $y \in \beta(V) \Longrightarrow y = \beta(x)$, some $x \in V$. This means $(\forall y \in \beta(V))\Phi(\overline{\alpha}(y), \delta(x))$, for some $\delta(x) \in S$. Thus $(\exists U \in \mathcal{U})(\forall x \in U)(\exists z \in S)\Phi(\overline{\alpha}(x), z)$.

"$\Longleftarrow$" Assume $(\exists U_1 \in \mathcal{U})(\forall x \in U_1)(\exists y \in S)\Phi(\overline{\alpha}(x), y)$. Assume that $\mathcal{U}$ is an ultrafilter on the set A, and let $B = A \times S$, a nonempty set. Let $\mathcal{V}$ be the ultrafilter expansion of the filter generated by the sets $\tilde{U} = \{(x, z) \mid x \in U \wedge \Phi(\overline{\alpha}(x), z)\}$, for $U \in \mathcal{U}$. By the assumption follows that each $\tilde{U}$ is not empty. The family of sets on the form $\tilde{U}$ has the finite intersection property since, given $U, V \in \mathcal{U}$, $\tilde{U} \cap \tilde{V} = \{(x, z) \mid x \in U \cap V \wedge \Phi(\overline{\alpha}(x), z)\}$ and $U \cap V \in \mathcal{U}$.

Consider the projections $\pi_1 : (B, \mathcal{V}) \to (A, \mathcal{U})$ and $\pi_2 : (B, \mathcal{V}) \to S$. $\pi_1 : (B, \mathcal{V}) \to (A, \mathcal{U})$ is continuous since $\pi_1^{-1}(U) = U \times S \supseteq \tilde{V} \in \mathcal{V}$, for $V = U \cap U_1 \in \mathcal{U}$. We have $(\exists V \in \mathcal{V})(\forall (x, z) \in V)\Phi(\overline{\alpha}(\pi_1(x, z)), \pi_2(x, z))$. By IH: $(B, \mathcal{V}) \Vdash {}^*\Phi(\overline{\alpha} \circ \pi_1, \pi_2)$. Since $\pi_1 : (B, \mathcal{V}) \to (A, \mathcal{U})$ is continuous and $\mathcal{U}$ an ultrafilter we have that π_1 is a covering. That is: there is $\beta : \mathcal{V} \to \mathcal{U}$ and $\delta \in {}^*S(\mathcal{V})$ $(\delta = \pi_2)$ such that $\mathcal{V} \Vdash {}^*\Phi(\overline{\alpha} \circ \beta, \delta)$.

□

From the above Theorem follows this external transfer principle:

Corollary 4.7. *For any L-formula Θ we have*

$$\Theta \textit{ is true } \Longleftrightarrow \mathcal{U} \Vdash {}^*\Theta,$$

for any $\mathcal{U} \in \mathbb{U}$.

We also have a result on the relation between truth in an ultrapower over $\mathcal{U}$ and $\mathcal{U} \Vdash$, which was promised in the last section.

Corollary 4.8. *For any structure ${}^*\mathbf{S}$, ultrafilter $\mathcal{U}$ and internal formula ${}^*\Theta(\overline{x})$ in ${}^*\mathbf{S}$ we have that*

$${}^*\mathbf{S}(\mathcal{U}) \models {}^*\Theta(\overline{\alpha}) \Longleftrightarrow \mathcal{U} \Vdash {}^*\Theta(\overline{\alpha}).$$

Proof. By Łoś's theorem we have ${}^*\mathbf{S}(\mathcal{U}) \models {}^*\Theta(\overline{\alpha})$ if and only if $\{x \mid \Theta(\overline{\alpha}(x))\} \in \mathcal{U}$. But this is equivalent to $(\exists U \in \mathcal{U})(\forall x \in U)\Theta(\overline{\alpha}(x))$. By Moerdijk's theorem the last statement is true if and only if $\mathcal{U} \Vdash {}^*\Theta(\overline{\alpha})$. □

There is also an internal transfer principle. If Ψ is a L-formula then let ${}^\sigma\Psi$ be the formula obtained by restricting all quantifiers in ${}^*\Psi$ to the standard objects. For example: ${}^\sigma((\forall x \in S)\, x > 0) = (\forall^{st} x \in {}^*S)\, x \,{}^*\!> {}^*0$.

Lemma 4.9. *Let $\Psi(x_1,\dots,x_n,y)$ be a L-formula with $x_1,\dots,x_n,y$ as its only free variables. Then the following is true in the internal logic of* $\mathrm{Sh}(\mathbb{U})$*:*

$$(\forall^{st}x_1\in{}^*T_1)\dots(\forall^{st}x_n\in{}^*T_n)[(\forall^{st}y\in{}^*S)^*\Psi(\overline{x},y)\to(\forall y\in{}^*S)^*\Psi(\overline{x},y)].$$

Proof. For any ultrafilter $\mathcal{U}$ and $t_1\in T_1,\dots,t_n\in T_n$, prove that

$$\mathcal{U}\Vdash(\forall^{st}y\in{}^*S)\,{}^*\Psi({}^*t_1,\dots,{}^*t_n,y)\to(\forall y\in{}^*S)\,{}^*\Psi({}^*t_1,\dots,{}^*t_n,y).$$

Assume $\mathcal{U}\Vdash(\forall^{st}y\in{}^*S)\,{}^*\Psi({}^*t_1,\dots,{}^*t_n,y)$. Then, for all $s\in S$, $\mathcal{U}\Vdash{}^*\Psi({}^*t_1,\dots,{}^*t_n,{}^*s)$. Since ${}^*\Psi$ is internal we have, by external transfer, that $(\forall s\in S)\,\Psi(t_1,\dots,t_n,s)$ is true. Then the *-transform of this holds in $\mathrm{Sh}(\mathbb{U})$, especially at $\mathcal{U}$, and thus $\mathcal{U}\Vdash(\forall y\in{}^*S)\,{}^*\Psi({}^*t_1,\dots,{}^*t_n,y)$. □

Theorem 4.10. *Let $\Psi(x_1,\dots,x_n)$ be a L-formula with $x_1,\dots,x_n$ as its only free variables.*

Then the following is true in the internal logic of $\mathrm{Sh}(\mathbb{U})$

$$(\forall^{st}x_1\in{}^*T_1)\dots(\forall^{st}x_n\in{}^*T_n)({}^*\Psi(\overline{x})\leftrightarrow{}^\sigma\Psi(\overline{x})).$$

Proof. This is proved by successive applications of the lemma, working from the inside and out. □

4.3. **Axiom of Choice.** In general, one says that a topos $\mathcal{E}$ satisfies the axiom of choice (AC) if every object P in $\mathcal{E}$ is *projective*, i.e. for any epimorphism $\eta:X\to P$ there is a morphism $\sigma:P\to X$ such that $\eta\circ\sigma=1$. Such a morphism $\sigma:P\to X$ is called a *section* of η.

The morphisms in $\mathrm{Sh}(\mathbb{U})$ are natural transformations. Epimorphisms in a Grothendieck topos is recognized by [8, Cor. III.7.5]:

$\eta:S\to T$ is an epimorphism if and only if for every $\mathcal{U}\in\mathbb{U}$ and $y\in T(\mathcal{U})$ there is a covering $\alpha:\mathcal{V}\to\mathcal{U}$ such that $T(\alpha)(y)\in\operatorname{Im}\eta_{\mathcal{V}}$.

To avoid confusion 1 will denote the terminal object in $\mathbb{U}$ and $\mathbf{1}$ the terminal object in $\mathrm{Sh}(\mathbb{U})$. The general topos theoretic AC does not hold in $\mathrm{Sh}(\mathbb{U})$. Let $\mathcal{U}_0$ be an ultrafilter in $\mathbb{U}$ not isomorphic to 1. Consider the representable sheaf $\mathbf{y}(\mathcal{U}_0)=\mathrm{Hom}_{\mathbb{U}}(\,\cdot\,,\mathcal{U}_0)$. By Propositions 2.10 and 2.11 we have that $\mathbf{y}(\mathcal{U}_0)(1)=\emptyset$.

Define a natural transformation $\tau:\mathbf{y}(\mathcal{U}_0)\to\mathbf{1}$ at $\mathcal{V}$

$$\tau_{\mathcal{V}}:\mathbf{y}(\mathcal{U}_0)(\mathcal{V})\to\mathbf{1}(\mathcal{V})$$

as $\tau_{\mathcal{V}}=\lambda x.0$, if $\mathcal{V}\ncong 1$, and $\tau_{\mathcal{V}}=\emptyset$, if $\mathcal{V}\cong 1$.

We must check that this definition gives a natural transformation. Take a morphism $\alpha:\mathcal{W}\to\mathcal{V}$ and show that

$$\tau_{\mathcal{W}}\circ\mathbf{y}(\mathcal{U}_0)(\alpha)=\mathbf{1}(\alpha)\circ\tau_{\mathcal{V}}.$$

If $\mathcal{W},\mathcal{V}\ncong 1$ then take $x\in\mathbf{y}(\mathcal{U}_0)(\mathcal{V})$. Now $\tau_{\mathcal{W}}(\mathbf{y}(\mathcal{U}_0)(\alpha)(x))=\tau_{\mathcal{W}}(x\circ\alpha)=0=\mathbf{1}(\alpha)(0)=\mathbf{1}(\alpha)(\tau_{\mathcal{V}}(x))$. If $\mathcal{V}\cong 1$ then $\mathbf{y}(\mathcal{U}_0)(\mathcal{V})$ is empty and the equality is true.

Now we will show that $\tau:\mathbf{y}(\mathcal{U}_0)\to\mathbf{1}$ is epi. Take a $\mathcal{V}\in\mathbb{U}$ and $0\in\mathbf{1}(\mathcal{V})$. Find a morphism $\alpha:\mathcal{W}\to\mathcal{V}$ such that $\mathbf{1}(\alpha)(0)\in\mathrm{Im}\tau_{\mathcal{W}}$. This is accomplished by taking $\alpha:\mathcal{W}\to\mathcal{V}$ such that $\mathcal{W}\ncong 1$ and such that there is also a morphism $\mathcal{W}\to\mathcal{U}_0$. Take $\mathcal{W}$ as the ultrafilter extension of the product filter $\mathcal{U}_0\times\mathcal{W}$ in $\mathbb{F}$. Then $\operatorname{Im}\tau_{\mathcal{W}}=\{0\}\ni 0=\mathbf{1}(\alpha)(0)$.

But the epimorphism $\tau : \mathbf{y}(\mathcal{U}_0) \to \mathbf{1}$ can not have a section $\sigma : \mathbf{1} \to \mathbf{y}(\mathcal{U}_0)$ since then we would have $\sigma_1 : \{0\} \to \emptyset$. Thus $\mathbf{1}$ is not projective, and $\mathrm{Sh}(\mathbb{U})$ does not have the axiom of choice.

Even if we do not have the full axiom of choice in the topos $\mathrm{Sh}(\mathbb{U})$ we at least have, by external transfer, the axiom of choice for internal formulas and sheaves on the form ${}^*S$ (note that this is not the same as the topos theoretic internal axiom of choice). We also have a partial result for some other situations based on the same result by Palmgren [14] for $\mathrm{Sh}(\mathbb{F})$.

Lemma 4.11. *Suppose that $\{\psi_x : \mathcal{U}_x \to \mathcal{U}\}_{x\in S}$ is a family of morphisms in $\mathbb{U}$. Then there exists an ultrafilter $\mathcal{V}$ and a family of morphisms $\{\gamma_x : \mathcal{V} \to \mathcal{U}_x\}_{x\in S}$ such that $\psi_x \circ \gamma_x = \psi_y \circ \gamma_y$, for all $x, y \in S$.*

Proof. The corresponding lemma (with the extra requirement that the morphisms are epi) is proved by Palmgren, but giving a filter $\mathcal{G}$ instead of the ultrafilter $\mathcal{V}$.

By expanding the filter $\mathcal{G}$ to an ultrafilter $\mathcal{V}$ and replacing the morphisms γ_x by $\gamma_x \circ \iota$, where $\iota : \mathcal{V} \to \mathcal{G}$ is given by the identity on the underlying set, we get the lemma. □

Theorem 4.12. *Let S be a non-empty set, and let $\mathbf{S} = \Delta(S)$. Let T and P be arbitrary sheaves on $\mathbb{U}$. For any subobject $R \subseteq \mathbf{S} \times T \times P$ the following is valid in $\mathrm{Sh}(\mathbb{U})$:*

$$\forall z[\forall x \exists y R(x, y, z) \to (\exists f \in T^{\mathbf{S}})\forall x R(x, f(x), z)].$$

Proof. Suppose $\mathcal{U} \Vdash (\forall x \in \mathbf{S})\exists y R(x, y, \zeta)$, where $\zeta \in P(\mathcal{U})$. Hence for each $x \in S$ there exists a morphism $\psi_x : \mathcal{U}_x \to \mathcal{U}$ and $y_x \in T(\mathcal{U}_x)$ such that

$$\mathcal{U}_x \Vdash R({}^*x, y_x, \zeta \cdot \psi_x).$$

By the Lemma there exists an ultrafilter $\mathcal{V}$ and a family of morphisms $\{\gamma_x : \mathcal{V} \to \mathcal{U}_x\}_{x\in S}$ such that $\psi_x \circ \gamma_x = \psi_y \circ \gamma_y$, for all $x, y \in S$. Let $x_0 \in S$ and put $\varepsilon = \psi_{x_0} \circ \gamma_{x_0}$. Then, for each $x \in S$

$$\mathcal{V} \Vdash R({}^*x, y_x \cdot \gamma_x, \zeta \cdot \varepsilon).$$

Let $[\mathbf{S} \to T]$ be the exponent of $\mathbf{S}$ and T in $\mathrm{Sh}(\mathbb{U})$. We find $f \in [\mathbf{S} \to T]$ such that $ev_{\mathcal{V}}(f, {}^*x) = y_x \cdot \gamma_x$. Thus

$$\mathcal{V} \Vdash R({}^*x, ev(f, {}^*x), \zeta \cdot \varepsilon),$$

for each $x \in S$. It follows that

$$\mathcal{V} \Vdash (\exists f \in [\mathbf{S} \to T])(\forall x \in \mathbf{S})R(x, ev(f, x), \zeta \cdot \varepsilon).$$

By local character this is forced already at $\mathcal{U}$. □

4.4. **Additional results.** In this section we collect some other theorems and principles true in $\mathrm{Sh}(\mathbb{U})$. Many of them follow almost directly from results regarding the topos $\mathrm{Sh}(\mathbb{F})$.

The idealization principle below is proved by Palmgren [13] for $\mathrm{Sh}(\mathbb{F})$.

Theorem 4.13. *Let ${}^*\Phi(x, y, z)$ be an internal formula. Take some sheaf ${}^*R$. Then the following is true for any $\mathcal{U} \in \mathbb{U}$: if, for all finite sets $S = \{s_1, \dots, s_n\} \subseteq R$ we have $\mathcal{U} \Vdash (\exists x \in {}^*T)\, {}^*\Phi(x, {}^*s_i, \alpha)$, for all $i = 1, \dots, n$, then $\mathcal{U} \Vdash (\exists x \in {}^*T)(\forall^{st} y \in {}^*R)\, {}^*\Phi(x, y, \alpha)$.*

Proof. Let ${}^*\Phi(x, y, z)$ be an internal formula.

Assume that for every $\mathcal{U} \in \mathbb{U}$ and finite $S = \{s_1, \dots, s_n\} \subseteq R$ we have

$$\mathcal{U} \Vdash (\exists x \in {}^*T)(\bigwedge_{i=1}^{n} {}^*\Phi(x, {}^*s_i, \alpha)).$$

For every $U \in \mathcal{U}$ and finite $S = \{s_1, \dots, s_n\} \subseteq R$ define a set V by

$$(u, x) \in V \iff u \in U \wedge x \in T \wedge \bigwedge_{i=1}^{n} \Phi(x, s_i, \alpha(u)).$$

By Moerdijk's theorem, $\bigwedge_{i=1}^{n} \Phi(x, s_i, \alpha(u))$ is true, for some $x \in T$, and all u in some $U' \in \mathcal{U}$. Since $\mathcal{U}$ is closed under intersections this gives that all sets V are non-empty. The family of sets V have the finite intersection property and can thus be extended to an ultrafilter $\mathcal{V}$. Now, we have a covering $\pi_1 : \mathcal{V} \to \mathcal{U}$ and an element $\pi_2 \in {}^*T(\mathcal{V})$.

It remains to prove that

$$\mathcal{V} \Vdash \forall^{st} y \in {}^*R\, {}^*\Phi(\pi_2, y, \alpha \circ \pi_1),$$

that is $\mathcal{V} \Vdash {}^*\Phi(\pi_2, {}^*s, \alpha \circ \pi_1)$, for all $s \in R$.

By Moerdijk's theorem $\mathcal{V} \Vdash {}^*\Phi(\pi_2, {}^*s, \alpha \circ \pi_1)$ if and only if there is an $V \in \mathcal{V}$ such that for all $(u, x) \in V$ $\Phi(x, s, \alpha(u))$. This is true for any set V from the initial family in $\mathcal{V}$ given by any $U \in \mathcal{U}$ and $S = \{s\}$. □

Standardization below is a restricted version of a separation axiom.

Theorem 4.14. *Let $\mathcal{U}$ be a ultrafilter in $\mathbb{U}$, ${}^*S$ a sheaf in $\mathrm{Sh}(\mathbb{U})$ and $\Phi(x, y)$ a formula. Then there exists a sheaf ${}^*T$ such that*

$$\mathcal{U} \Vdash \forall^{st} z\, (z \in {}^*T \leftrightarrow z \in {}^*S \wedge \Phi(z, \alpha)).$$

Proof. Let $\mathcal{U}$ be any ultrafilter, ${}^*S$ any sheaf and $\Phi(x, y)$ any formula.

Define the set $T = \{z \in S \mid \mathcal{U} \Vdash {}^*z \in {}^*S \wedge \Phi({}^*z, \alpha)\}$. Then

$$\mathcal{U} \Vdash \forall^{st} z\, (z \in {}^*T \leftrightarrow z \in {}^*S \wedge \Phi(z, \alpha)).$$

□

This is not separation since the condition on the elements in ${}^*T$ only holds true for the standard elements in ${}^*T$. It is not true, in general, that $z \in {}^*T \to \Phi(z)$ and neither, in general, that $\Phi(z) \wedge z \in {}^*S \to z \in {}^*T$.

4.5. **Factorising ultrafilters.** Considering Corollary 4.8 it is natural to ask whether $\mathcal{U} \Vdash$ is the same as truth in an ultrapower over the ultrafilter $\mathcal{U}$, not only for internal formulas but for formulas in the language ${}^*\mathrm{L} + \mathrm{St}$.

Inspecting Theorem 4.2 and Theorem 4.3 we see that the only problem for external ${}^*\mathrm{L} + \mathrm{St}$-formulas is in the treatment of the quantifiers. If we, for instance, had that, for any external ${}^*\mathrm{L} + \mathrm{St}$-formula Φ:

$$\mathcal{U} \Vdash (\forall y \in {}^*S)\Phi(\overline{\alpha}, y) \text{ iff for all } \gamma \in {}^*S(\mathcal{U}), \mathcal{U} \Vdash \Phi(\overline{\alpha}, \gamma) \qquad (\star)$$

then truth in $\mathrm{Sh}(\mathbb{U})$ at $\mathcal{U}$ and truth in an ultrapower over $\mathcal{U}$ would coincide. But as we shall see, $\mathcal{U}$ does not have $(\star)$ for all small sets S.

A sufficient condition for $\mathcal{U}$ to have $(\star)$ for S, is that $\mathcal{U}$ is factorising for S:

Definition 4.15. $\mathcal{U}$ is a *factorising filter* for S if for any morphism $\beta : \mathcal{V} \to \mathcal{U}$ and any $\delta : \mathcal{V} \to S$ there exists $\gamma : \mathcal{U} \to S$ such that $\gamma \circ \beta = \delta$.

Lemma 4.16. *If $\mathcal{U}$ is a factorising filter for S then $\mathcal{U}$ has $(\star)$ for S.*

Proof. Assume $\mathcal{U}$ is a factorising filter for S. Verify the equivalence in $(\star)$:
"$\Longrightarrow$": Clear. Take $[id] : \mathcal{U} \to \mathcal{U}$. Then for all $\delta \in {}^*S(\mathcal{U}) : \mathcal{U} \Vdash \Phi(\overline{\alpha}, \delta)$.
"$\Longleftarrow$": Take $\beta : \mathcal{V} \to \mathcal{U}$ and $\delta \in {}^*S(\mathcal{V})$. Show that $\mathcal{V} \Vdash \Phi(\overline{\alpha} \circ \beta, \delta)$.

Since $\mathcal{U}$ is a factorising filter for S there is $\gamma \in {}^*S(\mathcal{U})$ such that $\gamma \circ \beta = \delta$. That is $\Phi(\overline{\alpha} \circ \beta, \delta) \equiv \Phi(\overline{\alpha} \circ \beta, \gamma \circ \beta)$. By assumption in $(\star)$: $\mathcal{U} \Vdash \Phi(\overline{\alpha}, \gamma)$. Since $\Vdash$ is monotone we have that $\mathcal{V} \Vdash \Phi(\overline{\alpha} \circ \beta, \gamma \circ \beta)$, i.e. $\mathcal{V} \Vdash \Phi(\overline{\alpha} \circ \beta, \delta)$. □

But being a factorising filter for S is also, for some S, a necessary condition for $\mathcal{U}$ to have $(\star)$ for S.

Proposition 4.17. *If $(S,\mathcal{U})$ has $(\star)$ for S then $(S,\mathcal{U})$ is a factorising filter for S.*

Proof. Assume that $(S,\mathcal{U})$ has $(\star)$ for S.

Let $\Phi(x, y) \equiv (\exists^{st}\varphi \in {}^*(S \to S))\varphi(x) \mathrel{^*{=}} y$. That $\mathcal{U}$ is an ultrafilter gives that

$$\mathcal{U} \Vdash \Phi(\alpha, \gamma) \Longleftrightarrow \text{ there exists } f : S \to S \text{ such that } \mathcal{U} \Vdash {}^*f(\alpha) \mathrel{^*{=}} \gamma.$$

Observe that ${}^*f_{\mathcal{U}}(\alpha)(u) = (\lambda u.f(\alpha(u)))(u) = f(\alpha(u))$, i.e. ${}^*f(\alpha) = f \circ \alpha$. Take as $\alpha \in {}^*S(\mathcal{U})$ the function $\iota : (S,\mathcal{U}) \to S$, generated by the identity function $S \to S$. Now show: for any $\gamma \in {}^*S(\mathcal{U})$, $\mathcal{U} \Vdash \Phi(\iota, \gamma)$.

Given an arbitrary $\gamma \in {}^*S(\mathcal{U})$ take $f : S \to S$ as any total function extending γ. Then we have $f \circ \iota = \gamma$ on some $U \in \mathcal{U}$, so ${}^*f(\iota) \mathrel{^*{=}} \gamma$ and thus $\mathcal{U} \Vdash \Phi(\iota, \gamma)$. By $(\star)$ this means that $\mathcal{U} \Vdash (\forall y \in {}^*S)\Phi(\iota, y)$. That is for any $\beta : \mathcal{V} \to \mathcal{U}$ and any $\delta \in {}^*S(\mathcal{V})$ we have $\mathcal{V} \Vdash \Phi(\iota \circ \beta, \delta)$.

That is $\mathcal{V} \Vdash (\exists^{st}\varphi \in {}^*(S \to S))\varphi(\iota \circ \beta) \mathrel{^*{=}} \delta$. $\mathcal{V}$ is an ultrafilter so this means that there exists $f : S \to S$ such that $\mathcal{V} \Vdash {}^*f(\iota \circ \beta) \mathrel{^*{=}} \delta$. That is $f(\beta(u)) = \delta(u)$. But $f : S \to S$ also gives an element $[f] \in {}^*S(\mathcal{U})$.

So, for any $\beta : \mathcal{V} \to \mathcal{U}$ and $\delta \in {}^*S(\mathcal{V})$, there is a $[f] \in {}^*S(\mathcal{U})$ such that $[f] \circ \beta = \delta$. So (S, $\mathcal{U}$) is a factorising filter for S. □

Now, we will show that $(S,\mathcal{U})$ is not a factorising filter for S. We will use a reformulation of the property of being factorising.

Proposition 4.18. *$\mathcal{U}$ is a factorising filter for S if and only if for all $\beta : \mathcal{V} \to \mathcal{U}$ ${}^*S(\beta) : {}^*S(\mathcal{U}) \to {}^*S(\mathcal{V})$ is an epimorphism*

Proof. $\mathcal{U}$ is a factorising filter for S if and only if for all $\beta : \mathcal{V} \to \mathcal{U}$ and $\delta \in {}^*S(\mathcal{V})$ there exists a $\gamma \in {}^*S(\mathcal{U})$ such that $\gamma \circ \beta = \delta$. This is true if and only if for all $\beta : \mathcal{V} \to \mathcal{U}$ and $\delta \in {}^*S(\mathcal{V})$ there exists a $\gamma \in {}^*S(\mathcal{U})$ such that ${}^*S(\beta)(\gamma) = \delta$, since ${}^*S(\beta)(\gamma) = \gamma \circ \beta$. □

By Theorem 2.17 we know that no ultrafilter is maximal in the Rudin-Keisler ordering on a set S. This means that every ultrafilter on S is strictly covered by some other ultrafilter on S. Hence the proposition below gives that no ultrafilter on S is factorising for S.

Proposition 4.19. *Let $(S,\mathcal{U})$ be an ultrafilter and $(T, \mathcal{V})$ an ultrafilter covering $(S,\mathcal{U})$ such that $(T, \mathcal{V}) \not\cong (S,\mathcal{U})$. Then $\mathcal{U}$ is not a factorising filter for T.*

Proof. Assume that we have a covering $\alpha : (T,\mathcal{V}) \to (S,\mathcal{U})$ and that $(T,\mathcal{V}) \not\cong (S,\mathcal{U})$. Also assume that $\mathcal{U}$ is a factorising filter for T.

Since $(T,\mathcal{V}) \not\cong (S,\mathcal{U})$ we have that α is not a monomorphism, hence there are $\beta,\gamma : (R,\mathcal{W}) \to (T,\mathcal{V})$ such that $\alpha\circ\beta = \alpha\circ\gamma$ but $\beta \neq \gamma$. If we apply ${}^*T$ we get ${}^*T(\beta)\circ{}^*T(\alpha) = {}^*T(\gamma)\circ{}^*T(\alpha)$. But since $\mathcal{U}$ is a factorising filter for T, ${}^*T(\alpha)$ is an epimorphism and ${}^*T(\beta) = {}^*T(\gamma)$.

So ${}^*T(\beta)(x) = {}^*T(\gamma)(x)$, for all $x \in {}^*T(\mathcal{V})$, hence $x\circ\beta = x\circ\gamma$, for all $x : (T,\mathcal{V}) \to T$. Take $x = [id] : (T,\mathcal{V}) \to T$. Then $\beta = [id]\circ\beta = [id]\circ\gamma = \gamma$. Contradiction. □

5. Internal Set Theory

In this section let V be a set-theoretic universe (e.g. $V = V_\kappa$, where κ is an inaccessible cardinal) and consider only formulas in the language consisting of ${}^*\!\in$, ${}^*\!=$ and $\mathrm{St}^{{}^*V}$. We will show that ${}^*V$ models internal set theory (IST). IST was introduced by E. Nelson [10] and is an axiomatic approach to nonstandard analysis.

IST consists of the theory ZFC expanded with a new undefined unary predicate $\mathrm{St}(x)$ and three additional axioms: transfer (T), idealization (I) and standardization (S). We call a formula *internal* in case it does not involve the predicate $\mathrm{St}(x)$.

The axioms are:

(T) Let $\Theta(x,t_1,\ldots,t_n)$ be an internal formula with $x,t_1,\ldots,t_n$ as its only free variables. Then

$$\forall^{st}t_1\ldots\forall^{st}t_n(\forall^{st}x\,\Theta(x,t_1,\cdots,t_n) \to \forall x\,\Theta(x,t_1,\ldots,t_n))$$

is an axiom.

(I) Let $\Phi(x,y)$ be an internal formula. Then

$$\forall^{st\,fin}z\exists x\forall y\in z\,\Phi(x,y) \leftrightarrow \exists x\forall^{st}y\,\Phi(x,y)$$

is an axiom (here $\forall^{st\,fin}z\ldots$ means $\forall^{st}z\,(z\text{ finite}) \to \ldots$ where "z finite" is an abbreviation for some formula expressing finiteness, for instance that there is a bijection from z to $\{m\in\mathbb{N}\,|\,m<n\}$, for some $n\in\mathbb{N}$).

(S) Let $\Psi(z)$ be any formula. Then

$$\forall^{st}x\exists^{st}y\forall^{st}z(z\in y \leftrightarrow z\in x\wedge\Psi(z))$$

is an axiom.

We assume that V is a model of ZFC, and hence, by the external transfer principle previously established, we know that ${}^*V$ models ZFC (or, to be precise, *ZFC).

The new predicate $\mathrm{St}(x)$ is of course interpreted as $\mathrm{St}^{{}^*V}$. Now we will prove that ${}^*V$ models IST.

Theorem 5.1. *The transfer axiom holds in ${}^*V$.*

Proof. We have to prove that for any $\mathcal{U}\in\mathbb{U}$ and internal formula $\Theta(x,t_1,\ldots,t_n)$ in ${}^*V$ we have

$$\mathcal{U}\Vdash(\forall^{st}t_1\in{}^*V)\ldots(\forall^{st}t_n\in{}^*V)[(\forall^{st}x\in{}^*V)\Theta(x,\bar{t}) \to (\forall x\in{}^*V)\Theta(x,\bar{t})].$$

But this is already proved in Lemma 4.9. □

Suppose that there exists an unique x such that $\Theta(x)$, where $\Theta(x)$ is an internal formula with x as its only free variable. Then that x must be standard since by transfer (T) $\exists x\Theta(x) \Rightarrow \exists^{st} x\Theta(x)$. This means that all objects which can be uniquely described in set theory will be standard. In our model, for instance, ${}^*\mathbb{N}$, ${}^*\mathbb{R}$ and ${}^*\pi$ will be standard.

Theorem 5.2. *The idealization axiom holds in* ${}^*V$.

Proof. We have to prove that for any $\mathcal{U} \in \mathbb{U}$ and internal formula $\Phi(x, y, n)$ in ${}^*V$ we have

$$\mathcal{U} \Vdash (\forall^{st\,fin} z \in {}^*V)(\exists x \in {}^*V)(\forall y \in z)\, \Phi(x, y, \alpha) \leftrightarrow (\exists x \in {}^*V)(\forall^{st} y \in {}^*V)\, \Phi(x, y, \alpha).$$

The interesting direction to prove is from left to right. "z finite" in ${}^*V$ is an abbreviation for some suitable internal formula. Since a standard element in ${}^*V$ is on the form ${}^*z$, a finite standard element in the model is the *-transform of some finite set S. Now the result follows from Theorem 4.13. □

Theorem 5.3. *The standardization axiom holds in* ${}^*V$.

Proof. We have to prove that for any $\mathcal{U} \in \mathbb{U}$ and formula $\Psi(z, n)$ in ${}^*V$

$$\mathcal{U} \Vdash (\forall^{st} x \in {}^*V)\, (\exists^{st} y \in {}^*V)\, (\forall^{st} z \in {}^*V)\, (z \in y \leftrightarrow z \in x \wedge \Psi(z, \alpha)).$$

This is proved in Theorem 4.14. □

6. A short summary in Swedish

1993 presenterade I. Moerdijk en ny modell för icke-standard aritmetik i toposen av kärvar på en kategori av filter, Sh($\mathbb{F}$). Denna utvidgades senare av E. Palmgren till en modell för icke-standard analys. Modellen använder i första hand kärvarna ${}^*S$, som vid varje filter $\mathcal{F}$ ger den reducerade potensen av mängden S över $\mathcal{F}$, ${}^*S(\mathcal{F})$. I denna licentiat-avhandling fokuserar vi på kärvarna över delkategorin av ultrafilter, Sh($\mathbb{U}$). Kärvarna på formen ${}^*S$ är nu, vid ett ultrafilter $\mathcal{U}$, ultrapotensen av S över $\mathcal{U}$, ${}^*S(\mathcal{U})$.

Vi studerar den interna logiken i denna topos av kärvar, vilken är klassisk eftersom Sh($\mathbb{U}$) är en atomär topos. Vi visar att denna logik inte sammanfaller med logiken i någon av ultrapotenserna ${}^*S(\mathcal{U})$. Kategorin av ultrafilter har mycket gemensamt med ultrafiltren ordnade med Rudin-Keisler ordningen, till exempel har vi att $\mathcal{U} \leq \mathcal{V}$ om och endast om $\mathrm{Hom}_{\mathbb{U}}(\mathcal{V}, \mathcal{U}) \neq \emptyset$. I avhandlingen definierar vi Rudin-Keisler ordningen på Sh($\mathbb{U}$) och studerar de följder den har i vårt upplägg.

I avhandlingen studerar vi egenskaperna hos Sh($\mathbb{U}$). Vi visar två övergångs-principer: extern övergång, vilket är motsvarigheten till Łoś sats, och intern övergång. Vi visar att det topos-teoretiska urvalsaxiomet inte håller i Sh($\mathbb{U}$) men bevisar ett svagare resultat och också några andra egenskaper, liknande resultat som Palmgren visat för Sh($\mathbb{F}$).

Vi visar att toposen kan användas för att ge en modell för Nelsons interna mängdlära (IST). IST är en axiomatisk variant av icke-standard analys, där man utökar ZFC med ett odefinierat, unärt predikat St(x), för standard mängder, och axiom som relaterar standard och icke-standard mängder.

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