Inclusions and Non-Inclusions of Matching Archimedean and Laves Lattices

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Abstract

For all but one of the ordered pairs (G, H) of the matching graphs of the 19 Archimedean and Laves lattices, we solve the problem of deciding whether G is a subgraph of H or not.

1 Introduction

For two graphs G and H, let $G \subseteq H$ denote that G is isomorphic to a subgraph of H. We consider the problem of deciding whether $G \subseteq H$ or not, for all pairs of graphs in a class of 19 infinite graphs. Although a seemingly simple problem, some cases requiered considerable effort, and one case is left unresolved.

1.1 The Archimedean, Laves, and matching lattices

The *Archimedean lattices* are the graphs of vertices and edges of the regular tilings, which are vertex-transitive (i.e., for every pair of vertices, u and v, there is a graph isomorphism that maps u to v).

A regular tiling is a tiling of the plane which consists entirely of regular polygons; a polygon is regular if all side lengths are equal and all interior angles are equal. There are exactly 11 Archimedean lattices, as shown by Kepler, [6] (a modern proof can be found in [5, Ch. 2]); they are shown in Figure 1.

A notation for Archimedean lattices, which also serves as a prescription for constructing them, is given in Grünbaum and Shephard, [5]. Around any vertex, starting with the smallest polygon touching the vertex, list the number of edges of the successive polygons around the vertex. For convenience, an exponent is used to indicate that a number of successive polygons have the same size.

Since the Archimedean lattices are planar, each has a planar dual graph. The square lattice is self-dual, and the triangular and hexagonal lattices are a dual pair of graphs. The other 8 Archimedean lattices have dual graphs that are not Archimedean. These dual graphs are also shown in Figure 1. The dual

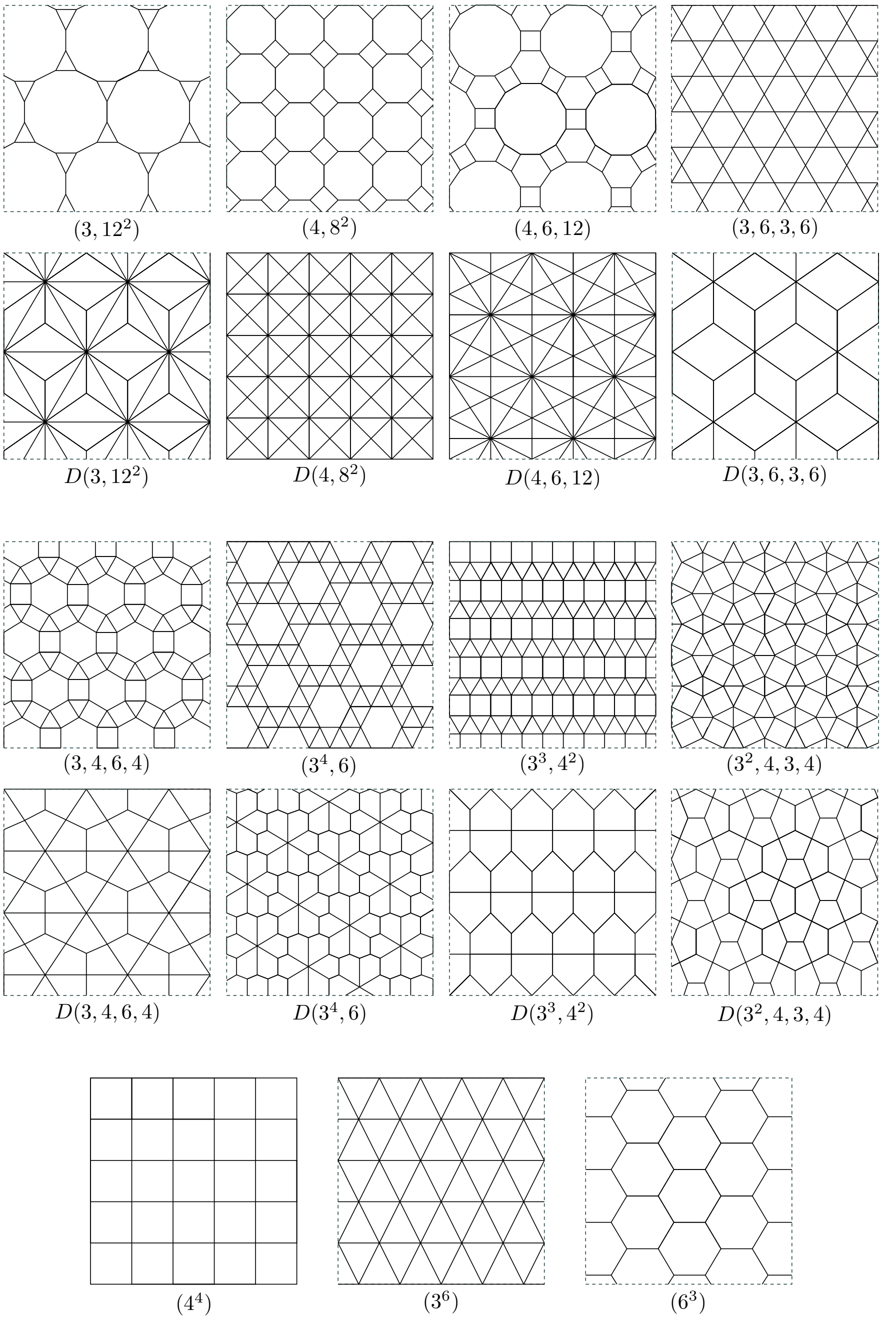

Figure 1: The 11 Archimedean lattices and their dual Laves lattices.

graph of an Archimedean lattice G will be denoted by $D(G)$. The duals of the Archimedean lattices have applications in crystallography, where they are called *Laves lattices* [9, 10]. The set of Archimedean lattices will be denoted by $\mathcal{A}$, the set of Laves lattices by $\mathcal{L}$.

A complete graph of order k is the graph with k vertices, such that each pair of vertices are connected, and will be denoted by K_k.

The *matching graph* of an infinite planar graph is the graph obtained by substituting each face with a complete graph of the corresponding order, i.e. every diagonal in every face is added. In Figure 2 the matching graph of the $(3, 6, 3, 6)$ lattice is shown.

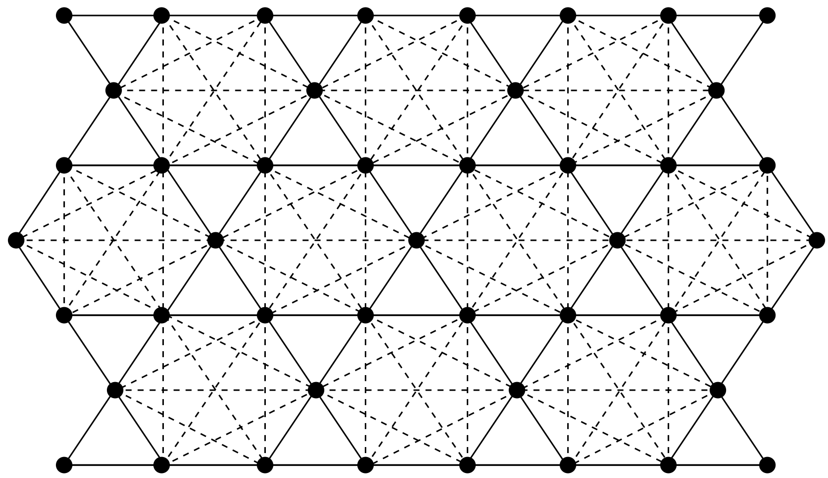

Figure 2: The matching graph of the $(3, 6, 3, 6)$ lattice. The added diagnonals are shown as dotted edges, vertices as black circles.

The matching graphs of infinite planar graphs are in general non-planar, the exception being triangulated graphs, which are their own matching graphs. We will denote the matching graph of G by $M(G)$, and the set of matching graphs of the Archimedean and Laves lattices by $\mathcal{M}$. A graph $G \in \mathcal{A} \cup \mathcal{L}$ will be called the base graph of its matching graph $M(G)$, and the edges of G the base edges of $M(G)$.

1.1.1 Some notation

If (u, v) is an edge, then u is adjacent to v and both are incident to the edge.

The degree $d(v)$ of a vertex is the number of edges incident to the vertex v. The minimal degree $\delta(G)$ of a graph with vertex set V is the minimum of the degrees of the vertices, $\delta(G) = \min\{d(v) : v \in V\}$. The maximal degree $\Delta(G)$ is the maximum of the degrees, $\Delta(G) = \max\{d(v) : v \in V\}$.

An n-cycle is a cycle of length n, and an n-vertex is a vertex of degree n. A polygon is defined to be a cycle with empty interior, and an n-gon is a polygon with n edges. We will frequently use the term triangle for 3-gons.

For $G \in \mathcal{M}$, let $K(G)$ denote the maximum value of k such that G contains a complete graph of order k. A complete graph $K_k \subseteq G$ is said to be *maximal in* G if K_k is not a subgraph of a complete graph of order $n > k$ in G. Let $\kappa(G)$ denote the minimum value of k, such that G contains a complete graph K_k of order k, maximal in G.

1.2 Summary and outline

The purpose of this paper is to determine which matching graphs of the lattices in $\mathcal{M}$ are subgraphs of others, and show which pairs are incomparable. This is done for all but one case; we have not been able to establish whether $M(3^3, 4^2)$ is a subgraph or not of $M(4, 6, 12)$. Since we show that $M(4, 6, 12)$ is not a subgraph of $M(3^3, 4^2)$, it is proved that for the matching Archimedean and Laves lattices, $\subseteq$ is anti-symmetric, and thus induces a partial order.

Remark 1.1. The relation $\subseteq$ is reflexive and transitive, but is not anti-symmetric in general for infinite graphs. An example of two non-isomorphic infinite tree graphs which are included in each other is given in [2, p. 231].

The corresponding problem for the Archimedean and Laves lattices themselves was completely solved by Parviainen and Wierman, [12]. That investigation was motivated by models from probability, combinatorics, and mathematical physics – especially percolation and self-avoiding walks. The results were applied to bond percolation critical probabilities in [24], and to connective constants for self-avoiding walks in [1].

1.2.1 Results

Table 1 and Figure 3 give concise summaries of the results. A "+" or "T" in the table indicates that the lattice at the top margin is a subgraph of the lattice at the left margin. Entries with "+" are illustrated by arrows in the figure, a Hasse diagram of the subgraph partial order (the undecided case is indicated by a dotted arrow). The arrow points from the subgraph, to the supergraph. Inclusions valid by transitivity are indicated by a "T" in the table (they are not shown in the figure). The unsolved case is indicated by a "?", and the other symbols indicates non-inclusions, the lemma or method used for the proof is used as the symbol.

1.2.2 Outline

This study is mainly motivated by site percolation, for which it extends the information given by the subgraph partial order for the Archimedean and Laves lattices, as explained in Section 2 below. In Section 3 the subgraph inclusions are demonstrated by figures. In Section 4 general techniques for showing non-inclusions are discussed, and in Section 5 the remaining non-inclusions, not handled by the general techniques, are proved.

	(3^6)	$D(4,8^2)$	$D(4,6,12)$	$D(3,12^2)$	$M(3^2,4,3,4)$	$M(3^3,4^2)$	$M(3^4,6)$	$M(D(3,6,3,6))$	$M(D(3,4,6,4))$	$M(4^4)$	$M(3,4,6,4)$	$M(3,6,3,6)$	$M(D(3^4,6))$	$M(D(3^3,4^2))$	$M(D(3^2,4,3,4))$	$M(6^3)$	$M(4,8^2)$	$M(4,6,12)$	$M(3,12^2)$
(3^6)		O	O	D	K	K	K	K	K	K	K	K	K	K	K	K	K	K	K
$D(4,8^2)$	O		O	D	K	K	K	K	K	K	K	K	K	K	K	K	K	K	K
$D(4,6,12)$	O	O		S	K	K	K	K	K	K	K	K	K	K	K	K	K	K	K
$D(3,12^2)$	+	S	S		K	K	K	K	K	K	K	K	K	K	K	K	K	K	K
$M(3^2,4,3,4)$	+	D	D	D		O	D	D	D	D	D	D	D	D	D	D	D	D	D
$M(3^3,4^2)$	+	D	D	D	O		D	D	D	D	D	D	D	D	D	D	D	D	D
$M(3^4,6)$	S	–	D	D	B	P		D	D	k	D	D	D	D	D	D	D	D	D
$M(D(3,6,3,6))$	+	P	S	+	P	S	K		O	O	K	K	D	K	K	K	D	D	D
$M(D(3,4,6,4))$	T	S	+	–	P	P	K	O		O	K	K	D	K	K	K	D	D	D
$M(4^4)$	T	+	D	D	+	+	K	O	O		D	D	D	D	D	D	D	D	D
$M(3,4,6,4)$	T	–	D	D	B	+	+	κ	κ	κ		D	D	D	D	D	D	D	D
$M(3,6,3,6)$	S	–	D	D	–	–	B	κ	κ	κ	B		D	D	D	D	D	D	D
$M(D(3^4,6))$	+	S	+	P	S	S	K	I	I	I	K	K		O	O	K	K	K	K
$M(D(3^3,4^2))$	T	S	S	P	P	+	K	I	I	I	K	K	O		O	K	D	D	D
$M(D(3^2,4,3,4))$	T	+	S	P	S	+	K	I	I	I	K	K	O	O		K	D	D	D
$M(6^3)$	T	T	T	P	T	T	T	–	+	+	+	P	I	I	I		D	D	D
$M(4,8^2)$	T	T	+	P	T	T	+	–	B	+	S	P	D	κ	κ	κ		D	D
$M(4,6,12)$	T	S	S	P	+	?	+	–	–	–	S	B	–	κ	–	–	B		D
$M(3,12^2)$	T	S	S	P	+	T	T	–	–	–	+	+	–	κ	–	–	κ	κ	

+	Inclusion
T	Inclusion by transitivity
?	Undecided
K	Maximal order
κ	Minimal order
B	Base graphs
I	Inconsistent complete graphs
D	Maximal degree
O	Same order of complete graphs
P	Special subgraphs
–	Non-inclusion by transitivity
S	Special case

Table 1: All inclusions and non-inclusions among the matching Archimedean and Laves lattices. Each entry indicates if the lattice listed at the top is included in the lattice listed at the left, and if not, indicates the reasoning that proves non-inclusion. The key at the right provides an interpretation for each symbol used in the table. A "+" indicates an inclusion which is demonstrated in the figures in Section 3. A "T" indicates an inclusion which is valid by transitivity. An "S" indicates that non-inclusion is proved in a special argument given in Section 5. All other symbols refer to a lemma or method, named by the symbol, in Section 4 for proving non-inclusion.

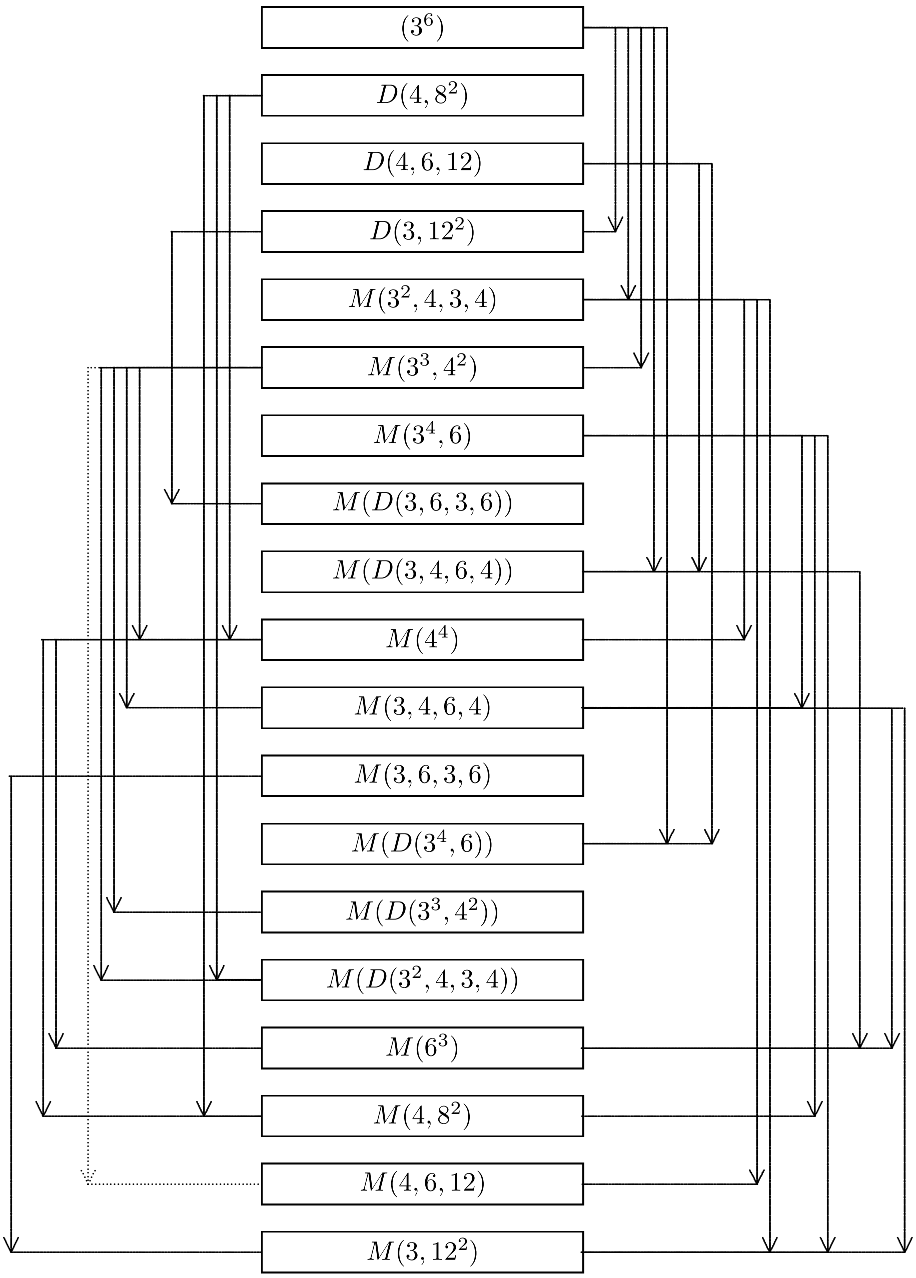

Figure 3: A Hasse diagram of the subgraph partial order. An arrow indicates that the lattice higher in the diagram is a subgraph of the lattice lower in the diagram. The dotted arrow represents the undecided inclusion. Additional subgraph relationships, valid by transitivity, are implied, but not shown.

2 Applications in Site Percolation

The two classical percolation models were introduced as models for the spread of fluid through a random medium. The medium is represented by an infinite connected locally finite graph.

In the site percolation model, each vertex is *open* with probability p, and fluid is permitted to flow through the subgraph induced by the set of open vertices. (In the bond percolation model, each edge of the graph is *open* to the flow of fluid with probability p.)

The key concept is the *critical probability*, or *percolation threshold*, denoted p_c, such that for $p < p_c$ there are almost surely no infinite connected components of open edges or vertices, and for $p > p_c$ there exists an infinite connected component with probability one. Considerable scientific interest focuses on percolation as a simple mathematical model for a phase transition, which is represented by the critical probability. See [13, 14] for descriptions of applications of percolation in engineering and physics. See Grimmett [4] for a complete discussion of the mathematical theory.

The exact bond or site critical probabilities are known only for a few graphs [7, 8, 17, 18], thus making it important to determine rigorous bounds for unsolved graphs [3, 19, 20, 21, 22, 23]. Many simulation studies have estimated critical probabilities of various graphs, in particular the Archimedean lattices [11, 15].

Let $p_c(G)$ denote the site percolation critical probability for the graph G. When comparing critical probabilities for two graphs, G and H, the values can be ordered if one graph is a subgraph of the other. If $G \subseteq H$, then $p_c(G) \geq p_c(H)$. Thus, for a given set of lattices, the subgraph partial order is a suborder of the order induced by the critical probabilities. Further information may possibly be gained from the subgraph partial order of the matching graphs, through the following relation, proved by Kesten, [8],

$$p_c(G) + p_c(M(G)) = 1.$$

Thus, if $M(G) \subseteq M(H)$, then $p_c(G) \leq p_c(H)$.

3 Inclusions

Figures 4 and 5 demonstrate the 27 subgraph inclusion relationships that are denoted by + entries in Table 1. These are the covering relationships in the Hasse diagram in Figure 3 for the subgraph ordering. Since the graphs are periodic, in each case sufficiently large induced subgraphs of the graphs are shown to demonstrate that the inclusion relationship can be extended throughout the infinite graphs.

Many of the inclusions were easily derived from the inclusions of Archimedean and Laves lattices, by the following lemma, also useful for proving non-inclusions. Write $G \hat{\subseteq} H$ if G can be obtained from H by removing only edges, and not vertices. (Any isolated vertices are by convention considered to be removed.)

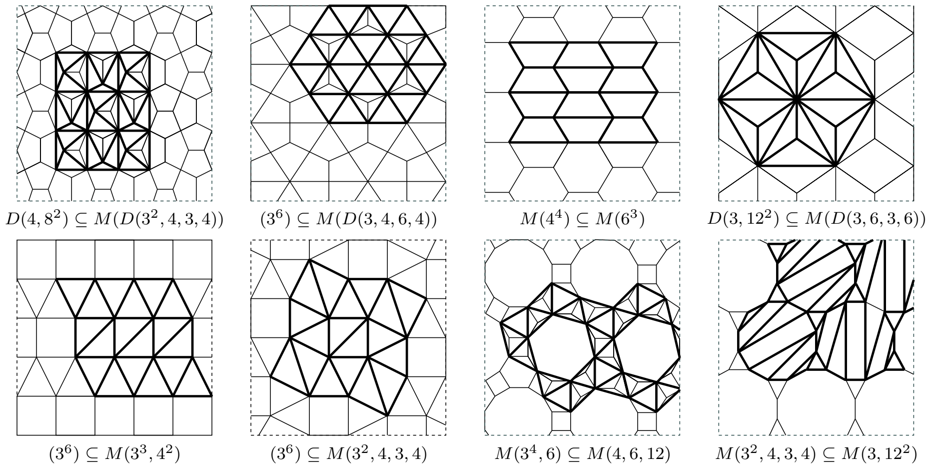

Figure 4: 8 Inclusions. The subgraphs are shown with bold edges. For the sake of clarity, we have omitted the diagonal edges in the complete subgraphs.

Lemma 3.1. $G \hat{\subseteq} H \iff M(H) \hat{\subseteq} M(G)$

Proof. All four graphs have identical vertex sets. Assume that $G \hat{\subseteq} H$. Each face in G is either also a face in H, or the union of incident faces of H. Thus, each complete graph of order k in $M(H)$, is either a complete graph of order k in $M(G)$, or a subgraph of a complete graph of order greater than k.

The converse implication is similar. □

This raises the questions whether there exists two graphs $G, H \in \mathcal{A} \cup \mathcal{L}$, such that $G \subseteq H$, but $M(H) \not\subseteq M(G)$, or such that $G \not\subseteq H$, but $M(H) \subseteq M(G)$? A negative answer to the second question would imply that this study of inclusions of the matching lattices would not extend the information about site percolation thresholds given by the inclusions of the Archimedean and Laves lattices themselves.

However, the answer to both question are positive. There are 27 pairs of graphs $G, H \in \mathcal{A} \cup \mathcal{L}$, such that $G \subseteq H$, but $M(H) \not\subseteq M(G)$, for instance,

$$(4,8^2) \subseteq D(3,6,3,6) \text{ but } M(D(3,6,3,6)) \not\subseteq M(4,8^2).$$

For the other direction, there are four cases,

$$D(3^3,4^2) \not\subseteq (3^3,4^2) \text{ but } M(3^3,4^2) \subseteq D(3^3,4^2),$$

$$(6^3) \not\subseteq (3,4,6,4) \text{ but } M(3,4,6,4) \subseteq M(6^3),$$

$$D(3,4,6,4) \not\subseteq (3^6) \text{ but } (3^6) \subseteq M(D(3,4,6,4)), \text{ and}$$

$$D(3^4,6) \not\subseteq D(4,6,12) \text{ but } D(4,6,12) \subseteq M(D(3^4,6)).$$

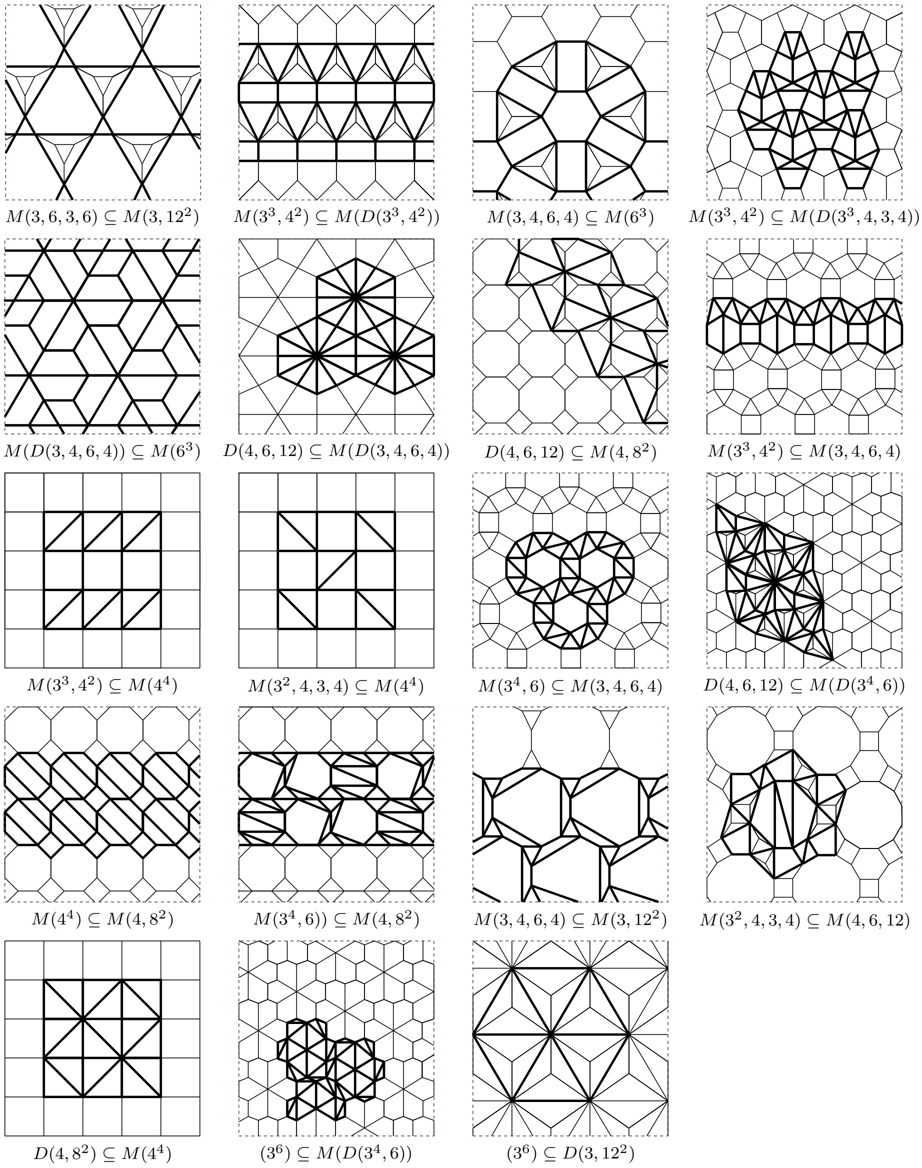

Figure 5: 19 Inclusions. The subgraphs are shown with bold edges. For the sake of clarity, we have omitted the diagonal edges in the complete subgraphs.

4 Non-Inclusions

Consider the possible inclusion $M(G) \subseteq M(H)$, where $G, H \in \mathcal{A} \cup \mathcal{L}$. Any given vertex v in $M(H)$ may be removed if and only if it has degree 3 in the base graph H, since removing it gives a polygon of size equal to the degree of v in the base graph H, and polygons of size larger than 3 do not exist in graphs in $\mathcal{M}$.

A connected set of vertices may similarly be removed only if the set is in the interior of a 3-cycle. This also applies to edges in the base graph H. Unless at least one of the incident vertices to the edge is in the interior of a 3-cycle, the edge may not be removed.

This simplifies the arguments for the cases where the candidate supergraph $M(H)$'s base graph H only have vertices of degree greater than 3, since we then know that the base edges and vertices cannot be removed.

4.1 General methods

By the reasoning above and by Lemma 3.1, it follows that non-inclusions can in some cases be deduced from non-inclusions for the base graphs.

Lemma B (Base graphs). *Assume that $G, H \in \mathcal{A} \cup \mathcal{L}$, and that $H \not\subseteq G$.*
i) If $\delta(H) > 3$, then $M(G) \not\subseteq M(H)$.
ii) If $\kappa(G) > 3$, then $M(G) \not\subseteq M(H)$.

Lemma B i) gives the non-inclusions

$$M(3^2, 4, 3, 4) \not\subseteq M(3^4, 6), M(3, 4, 6, 4),$$

$$M(3^4, 6), M(3, 4, 6, 4) \not\subseteq M(3, 6, 3, 6), \text{ and } M(3, 6, 3, 6) \not\subseteq M(4, 6, 12).$$

Lemma B ii) gives the non-inclusions

$$M(D(3, 4, 6, 4)) \not\subseteq M(4, 8^2) \text{ and } M(4, 8^2) \not\subseteq M(4, 6, 12).$$

As the degree of a vertex cannot be raised by removing edges, we have the following lemma.

Lemma D (Maximum degree). *If G and H are two graphs, and $\Delta(G) > \Delta(H)$, then $G \not\subseteq H$.*

The order of the complete subgraphs gives two other useful lemmas.

Lemma K (Maximum order). *If $G, H \in \mathcal{M}$, and $K(G) > K(H)$, then $G \not\subseteq H$.*

Proof. Complete graphs of higher order than $K_{K(H)}$ cannot be obtained in H by removing edges or vertices, so $G \not\subseteq H$. □

Lemma κ (Minimum order). *If $G, H \in \mathcal{M}$, such that $\kappa(G) > \kappa(H)$, then $G \not\subseteq H$.*

Proof. Assume $G \subseteq H$. Since H does not have polygons of size greater than 3, and $\kappa(H) \geq 3$, each complete graph of order n in H must include a complete graph of order k, $3 \leq k \leq n$, in G. Therefore $\kappa(G) \leq \kappa(H)$. □

If two graphs G and H in $\mathcal{M}$ consist only of complete graphs of order k_G and k_H, respectively, there are values of k_G and k_H such that G cannot be a subgraph of H, although $k_G < k_H$.

Lemma I (Inconsistent complete subgraphs).
If $G, H \in \mathcal{M}$, $G \subseteq H$ and $K(G) = \kappa(G) = k_G$, $K(H) = \kappa(H) = k_H$, then

$$k_H + 2(n-1) = nk_G, \text{ for some integer } n \geq 1.$$

Proof. Each complete subgraph in H of order k_H must be divided into n complete graphs of order k_G. This is possible if and only if $(k_H + 2(n-1))/n = k_G$. □

Example 4.1. With $k_G = 4$, and $k_H = 5$, this gives the non-inclusions

$$M(D(3,6,3,6)), M(D(3,4,6,4)), M(4^4)$$

$$\not\subseteq M(D(3^4,6)), M(D(3^3,4^2)), M(D(3^2,4,3,4)).$$

Example 4.2. With $k_G = 5$, and $k_H = 6$, we get

$$M(D(3^4,6)), M(D(3^3,4^2)), M(D(3^2,4,3,4)) \not\subseteq M(6^3).$$

Lemma O (Same order of complete graphs).
i) Let $G, H \in \mathcal{M}$, $G \neq H$. If $\kappa(G) = K(G) = \kappa(H) = K(H) > 3$, then $G \not\subseteq H$.
ii) If $G, H \in \{3^6, D(4,8^2), D(4,6,12)\}$ then $G \not\subseteq H$.

Proof. i) Removing any edge or a vertex from H gives a graph H', with $\kappa(H') < \kappa(H)$. By Lemma κ, $G \not\subseteq H'$. But if $G \subset H$, at least one edge must be removed.
ii) Removing any set of edges and/or vertices from H gives a polygon of size greater than 3, so $G \not\subseteq H$. □

Remark 4.1. A straightforward extension of Lemma O shows that $M(3^3,4^2)$ and $M(3^2,4,3,4)$ are incomparable.

Remark 4.2. Note that $D(3,12^2)$ could not be included in the group of fully-triangulated lattices in (ii) above, because vertices and edges can be deleted to obtain different lattices which are still fully-triangulated. In fact, $D(3,12^2)$ does contain (3^6) as a subgraph!

Some non-inclusions can be deduced by transitivity. By negating the statement

$$F \subseteq G \text{ and } G \subseteq H \Longrightarrow F \subseteq H,$$

we get the following lemma.

Lemma – (Non-inclusions by transitivity).
i) If $F \subseteq G, F \nsubseteq H$, then $G \nsubseteq H$.
ii) If $G \subseteq H, F \nsubseteq H$, then $F \nsubseteq G$.

Example 4.3.

$$D(3,12^2) \nsubseteq M(4,8^2) \text{ and } D(3,12^2) \subseteq M(D(3,6,3,6)).$$

By Lemma – i), $M(D(3,6,3,6)) \nsubseteq M(4,8^2)$.
Example 4.4.

$$D(4,8^2) \nsubseteq M(3,12^2) \text{ and } M(3,4,6,4), M(3,6,3,6) \subseteq M(3,12^2).$$

By Lemma – ii), $D(4,8^2) \nsubseteq M(3,4,6,4), M(3,6,3,6)$.

4.2 Method P (Special subgraphs).

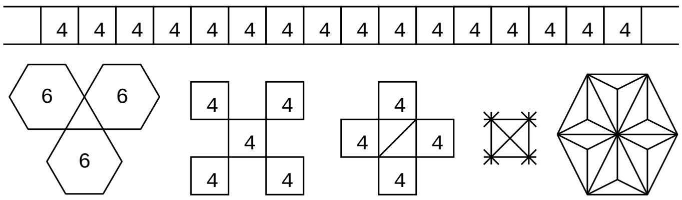

Figure 6: Subgraphs used by the criterion "Special subgraphs".

In several cases, where none of the above methods apply, it is possible to find a "small" subgraph in the candidate subgraph G, which easily can be seen not to exist in the candidate supergraph H. Therefore $G \nsubseteq H$. These cases are collected here. The "small" subgraphs used are shown in Figure 6.
Example 4.5. In $M(D(3,4,6,4))$, $M(3^4,6)$, $M(3,6,3,6)$, there does not exist a bi-infinite sequence of K_4's, such that each K_4 is incident to two other at opposite edges. This graph is a subgraph of $M(3^3,4^2)$. Therefore,

$$M(3^3,4^2) \nsubseteq M(D(3,4,6,4)), M(3^4,6), M(3,6,3,6).$$

Example 4.6. It is not possible to get a triangle which is incident to three K_6's by removing edges and vertices from either $M(6^3)$ or $M(4,8^2)$. This structure exists in $M(3,6,3,6)$, so

$$M(3,6,3,6) \nsubseteq M(6^3), M(4,8^2).$$

Example 4.7. By removing edges and vertices from $M(D(3,6,3,6))$, or from $M(D(3,4,6,4))$, it is not possible to get a K_4, which at each vertex is adjacent to 4 other, mutually disjoint, K_4's. This structure exists in $M(3^2,4,3,4)$. Therefore

$$M(3^2,4,3,4) \not\subseteq M(D(3,6,3,6)), M(D(3,4,6,4)).$$

Example 4.8. Two incident triangles, with the remaining four edges incident to K_4's, which exist in $M(3^2,4,3,4)$, cannot be found in $M(D(3^3,4^2))$.

$$M(3^2,4,3,4) \not\subseteq M(D(3^3,4^2)).$$

Example 4.9. In $D(4,8^2)$ there exists a 4-cycle of 8-vertices, surrounding another vertex, which in turn is adjacent to all 4 vertices in the cycle. This does not exist in $M(D(3,6,3,6))$.

$$D(4,8^2) \not\subseteq M(D(3,6,3,6))$$

Example 4.10. In none of $M(D(3^4,6^4))$, $M(D(3^3,4^2))$, $M(D(3^2,4,3,4))$, $M(6^3)$, $M(4,8^2)$, $M(3,12^2)$ or $M(4,6,12)$ can we find six edge-adjacent triangles with a vertex v in common, each triangle with a vertex in the interior, adjacent to all three vertices in the triangle.

$$D(3,12^2) \not\subseteq M(D(3^4,6^4)), M(D(3^3,4^2)), M(D(3^2,4,3,4)),\\ M(6^3), M(4,8^2), M(4,6,12), M(3,12^2)$$

5 Special Cases

5.1 $D(3,12^2) \not\subseteq D(4,6,12)$, $D(4,8^2)$ and $D(4,6,12) \not\subseteq D(3,12^2)$

These cases are proved in [12].

5.2 $D(4,8^2) \not\subseteq M(4,6,12)$

First note that vertices may be removed from $M(4,6,12)$, since they have degree 3 in the base graph. However, base edges may not be removed without removing exactly one of the incident vertices, since this would create polygons of size larger than 3, and $D(4,8^2)$ is triangulated.

Consider a K_{12} in $M(4,6,12)$, and assume that n vertices are removed from this K_{12}. Note that at most 6 vertices may be removed.

The boundary of the constructed K_{12-n} is a $12-n$-cycle, which will be central to the proof, and have the following properties.

Property A. There exists a path, of length at most 6, between any pair of vertices of distance 2 in the cycle, such that only the first and last vertices in the path are in the $(12-n)$-gon.

Define the minimal out-degree of a vertex in the cycle as the minimum number of incident edges with the other endpoint outside the complete subgraph, if no polygons of size 4 or larger exist in the subgraph of $M(4,6,12)$.

Property B. Each vertex have minimal out-degree at least 1. (If $n = 0$ the vertices have minimal out-degree 1, if a vertex is removed, the adjacent vertices minimal out-degree increases by one.)

Property C. There are no vertices in the interior of the cycle.

These properties will continue to hold as further vertices and edges are removed from $M(4,6,12)$, as long as no polygons of size 4 or larger are created.

In a series of lemmas we will show that no vertices can be removed from $M(4,6,12)$, if $D(4,8^2) \subseteq M(4,6,12)$. The first lemma states that it is only possible to remove an even number of vertices from one K_{12}. The proof is not hard, but rather long, and therefore omitted.

Lemma 5.1. *For $k = 7,9,11$, there exist no k-cycle in $D(4,8^2)$ that fulfills properties A, B and C.*

The next three lemmas show, in turn, that we cannot remove 6, 4, or 2 vertices from one K_{12}. The proofs are based on the following observation. Assume that n vertices are removed from one K_{12} in $M(4,6,12)$, giving a cycle as above with minimal out-degree sequence $\{m_1, m_2, \ldots, m_{12-n}\}$, and that $D(4,8^2) \subseteq M(4,6,12)$. Then a $12-n$-cycle that fulfills properties A, B and C must exist in $D(4,8^2)$, and have out-degree sequence $\{d_1, d_2, \ldots, d_{12-n}\}$, with $d_i \geq m_i$.

Lemma 5.2. $n = 6$ *vertices cannot be removed.*

Proof. There is only one way of removing six vertices from the K_{12}, resulting in a 6-cycle with minimal out-degree sequence $\{3,3,3,3,3,3\}$. There does not exist a 6-gon with out-degree sequence at least $\{3,3,3,3,3,3\}$ in $D(4,8^2)$. □

Lemma 5.3. $n = 4$ *vertices cannot be removed.*

Proof. There are two 8-cycles in $D(4,8^2)$ that fulfill properties A, B and C (shown in Figure 7). They have out-degree sequences $\{1,3,2,5,1,3,2,5\}$ and $\{1,2,5,1,5,1,5,2\}$.

There are 10 ways of removing 4 vertices from the K_{12}. These get minimal out-degree sequence

1. $\{1,1,1,2,3,3,3,2\}$,
2. $\{1,1,2,3,3,2,2,2\}$ (2 ways),
3. $\{1,2,3,3,2,1,2,2\}$,
4. $\{1,1,2,3,2,2,3,2\}$ (2 ways),

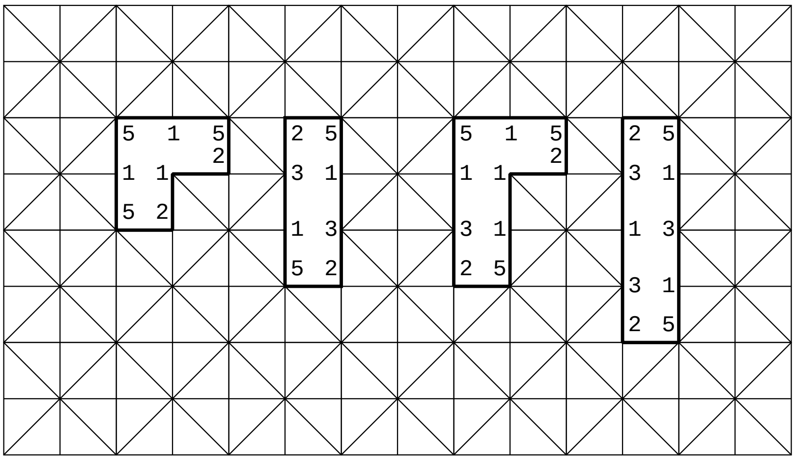

Figure 7: The two 8-cycles and the two 10-cycles in $D(4, 8^2)$ that fulfill properties A, B and C.

5. $\{1, 2, 3, 2, 2, 2, 2, 2\}$ (2 ways),
6. $\{1, 2, 2, 2, 3, 2, 2, 2\}$, and
7. $\{2, 2, 2, 2, 2, 2, 2, 2\}$.

All of these are inconsistent with the degree sequences of the 8-gons in $D(4, 8^2)$. □

Lemma 5.4. $n = 2$ *vertices cannot be removed.*

Proof. There are two 10-gons in $D(4, 8^2)$ that fulfills properties A, B and C (shown in Figure 7). They have out-degree sequence $\{2, 3, 1, 3, 2, 5, 1, 3, 1, 5\}$ and $\{1, 1, 5, 3, 1, 5, 1, 5, 2\}$.

There are 7 ways of removing 2 vertices from the K_{12}. These get minimal out-degree sequence

1. $\{2, 3, 2, 1, 1, 1, 1, 1, 1, 1\}$,
2. $\{2, 2, 2, 2, 1, 1, 1, 1, 1, 1\}$ (2 ways) ,
3. $\{2, 2, 1, 2, 2, 1, 1, 1, 1, 1\}$,
4. $\{2, 2, 1, 1, 2, 2, 1, 1, 1, 1\}$ (2 ways),
5. $\{2, 2, 1, 1, 1, 2, 2, 1, 1, 1\}$.

Of these, 3-5 are consistent with the 10-cycle in $D(4, 8^2)$ with degree sequence $\{2, 3, 1, 3, 2, 5, 1, 3, 1, 5\}$, and 4 and 5 are consistent with the 10-cycle in $D(4, 8^2)$ with degree sequence $\{1, 1, 5, 3, 1, 5, 1, 5, 2\}$.

This gives a total of 8 possible alignments, which are shown in Figure 8. Note that adjacent to the edges between vertices with minimal out-degree 2, there must be a triangle, which must be edge-adjacent to a further triangle, which in turn must have one edge in a 10- or 12-cycle, obeying properties A, B and C, at the specified positions. However, no such cycles exist.

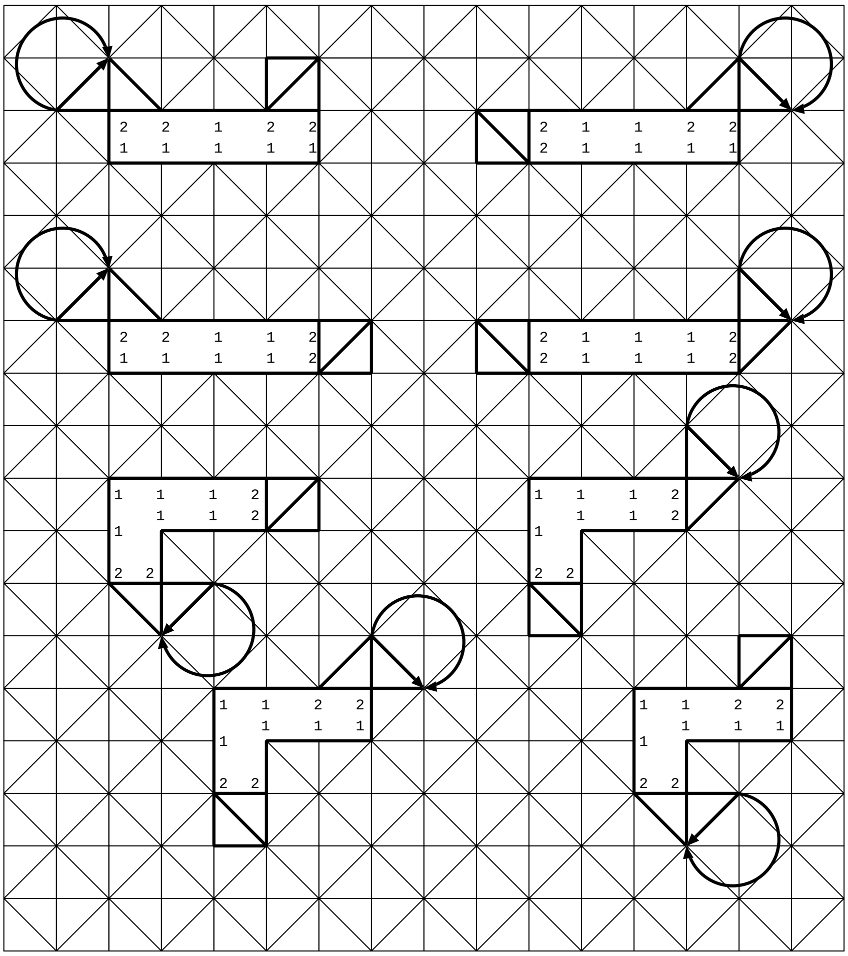

Figure 8: Possible alignments of 8-cycles, with triangles at the required places. The arrows indicates where there should be further 10- or 12-cycles.

□

To conclude the proof, we must show that by only removing edges from $M(4,6,12)$, $D(4,8^2)$ cannot be obtained. This is equivalent to showing that $(4,6,12)$ cannot be obtained from $D(4,8^2)$ by only edge removals.

Lemma 5.5. $(4,6,12)\hat{\not\subseteq}D(4,8^2)$.

Proof. Note that properties A, B and C must hold for the 12-cycles in $D(4,8^2)$ that are to be used as 12-gons in $(4,6,12)$. There are 4 such 12-cycles in $D(4,8^2)$, see Figure 9.

Only for one of these cycles, it is possible to find 4-gons and 6-gons around this 12-cycle in $D(4,8^2)$, to get a configuration consistent with $(4,6,12)$, see Figure 9. This configuration of one 12-gon, six 4-gons and six 6-gons cannot be continued to get $(4,6,12)$.

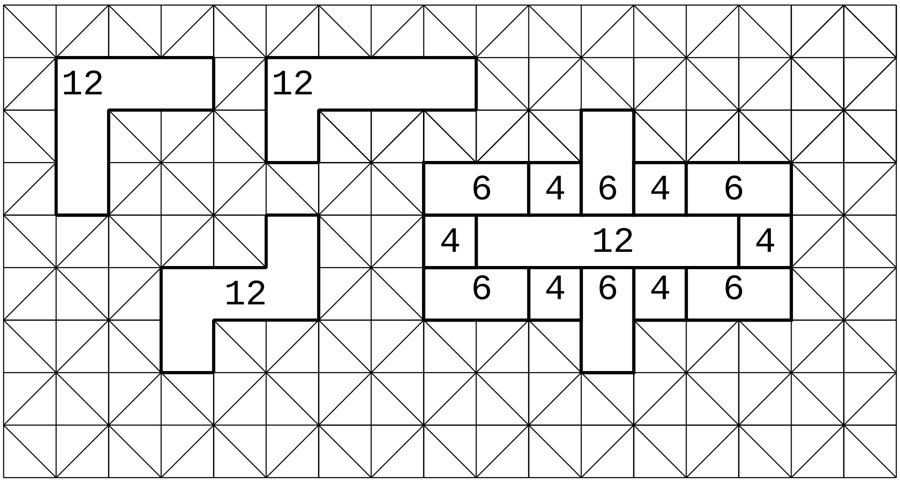

Figure 9: Possible 12-cycles in $D(4,8^2)$ that fulfills properties A, B and C.

□

5.3 $D(4,6,12) \not\subseteq M(4,6,12)$

This case is similar to the above, but considerable easier. There exists only 2 k-gons in $D(4,6,12)$ which obeys properties A, B and C, both have length $k=8$, and are shown in Figure 10. They have out-degree sequences

1. $\{1,3,1,3,1,4,5,4\}$, and
2. $\{1,3,1,2,2,5,2,3\}$.

Both are inconsistent with all 7 possible minimal out-degree sequences that can be obtained by removing 4 vertices from a K_{12} in $M(4,6,12)$.

5.4 $M(3^2,4,3,4) \not\subseteq M(D(3^2,4,3,4))$

First consider removing one vertex from $M(D(3^2,4,3,4))$ with degree 3 in the base graph. This gives a triangle, which is adjacent to 3 K_4's. From one of these

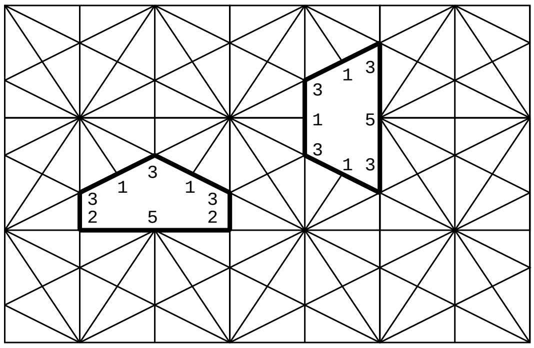

Figure 10: Possible alignments of 8-cycles.

K_4's we must either remove one further vertex, or divide it into 3 triangles. In all cases we get a vertex at which 3 triangles meet, which does not occur in $M(3^2, 4, 3, 4)$. Since vertices cannot be removed from $M(D(3^2, 4, 3, 4))$, and $D(3^2, 4, 3, 4) \not\subseteq (3^2, 4, 3, 4)$, it follows that $M(3^2, 4, 3, 4) \not\subseteq M(D(3^2, 4, 3, 4))$.

5.5 $M(3, 4, 6, 4) \not\subseteq M(4, 6, 12)$

Note that removing any vertex from $M(4, 6, 12)$ gives two edge-adjacent triangles, which do not exist in $M(3, 4, 6, 4)$. Therefore the K_6's in $M(4, 6, 12)$ must be used as either K_6's, or divided into one K_4 and two K_3's.

Assume that all K_6's in $M(4, 6, 12)$ are used as a K_6's in $M(3, 4, 6, 4)$. This implies that only edges from the K_{12}'s in $M(4, 6, 12)$ may be removed, which cannot give $M(3, 4, 6, 4)$.

Assume therefore that one K_6 is used to get two K_3's and one K_4. The resulting K_4 should be edge-adjacent to two K_6's, but will be edge-adjacent to one K_{12} and one K_4.

5.6 $M(3, 4, 6, 4) \not\subseteq M(4, 8^2)$

Consider constructing a K_6 from a K_8 in $M(4, 8^2)$. Since the K_6's in $M(3, 4, 6, 4)$ are adjacent to only K_4's, the K_8 must be divided into one K_6 and one K_4. This can be done in two ways, as shown in Figure 11. Neither case can be continued to get $M(3, 4, 6, 4)$.

The first case fails since the K_4 must be adjacent to another K_6 at the position marked by 6. The second fails since the K_4 must be adjacent to two triangles at the positions marked by 3. These triangles must each be adjacent to two further K_4's, which is impossible to achieve.

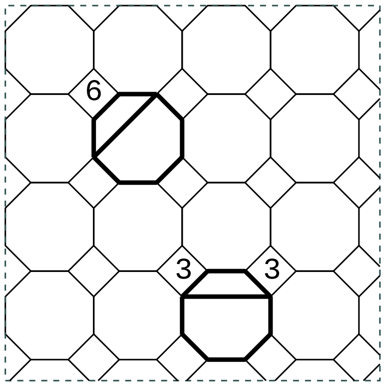

Figure 11: Possible ways of dividing a K_8 in $M(4,8^2)$ into one K_4 and one K_6.

5.7 $D(4,8^2) \not\subseteq M(D(3^3,4^2))$

We first note that the 9-vertices in $M(D(3^3,4^2))$ cannot be used to get 8-vertices in $D(4,8^2)$. Remove one edge incident to a 9-vertex in $M(D(3^3,4^2))$, to get the subgraph shown to left in Figure 12. Denote this subgraph F.

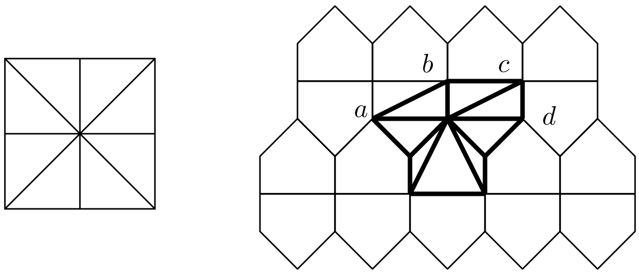

Figure 12: The vertex a is not connected to c with a path of length 2, edge-disjoint with F, nor is b connected to d with a path of length 2, edge-disjoint with F.

No matter which edge we remove, there will always be a path of 4 vertices $\{a,b,c,d\}$ on the boundary of F such that neither a is connected to c with a path of length 2, edge-disjoint with F, nor b connected to d with a path of length 2, edge-disjoint with F, which is required. An example is shown to the right in Figure 12

Therefore the 12-vertices in $M(D(3^3,4^2))$ must be used as 8-vertices, but these are only adjacent to 2 other 12-vertices, and not 4 as required.

5.8 $M(3^2, 4, 3, 4)$ and $M(3^3, 4^2) \not\subseteq M(D(3^4, 6))$

Consider a vertex of degree 18 in $M(D(3^4, 6))$. Since the vertex cannot be removed, 2 K_4's and 3 triangles must meet at the vertex. We must therefore remove 11 edges, which is impossible to do in a way that is consistent with either $M(3^2, 4, 3, 4)$ or $M(3^3, 4^2)$.

5.9 $D(4, 8^2) \not\subseteq M(D(3^4, 6))$

First note that since the 18-vertices in $M(D(3^4, 6))$ are not in the interior of 4-cycles, they cannot be used as 4-vertices in $D(4, 8^2)$.

A case by case analysis (11 cases in total) shows that the 9-vertices in $M(D(3^4, 6))$ cannot be used to get 8-vertices in $D(4, 8^2)$. But the 18-vertices are not adjacent to any other 18-vertex – in $D(4, 8^2)$ the 8-vertices are adjacent to 4 other, so $D(4, 8^2) \not\subseteq M(D(3^4, 6))$

5.10 $D(4, 8^2) \not\subseteq M(3, 12^2)$

Consider an 8-vertex in $D(4, 8^2)$. This has 4 adjacent 4-vertices, and this group of 5 vertices is in the interior of an 8-cycle, such that every other vertex is adjacent to the 8-vertex in the interior.

By choosing a vertex in $M(3, 12^2)$ as an 8-vertex in $D(4, 8^2)$, and considering the possible choices for the adjacent 4-vertices, it can be shown that this subgraph (shown in Figure 13) does not exist in $M(3, 12^2)$.

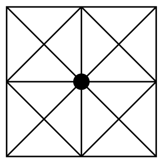

Figure 13: A subgraph of $D(4, 8^2)$ which do not exists in $M(3, 12^2)$.

5.11 $D(4, 6, 12) \not\subseteq M(3, 12^2)$

Consider any vertex v in $M(3, 12^2)$, and assume that this vertex will by edge removals be transformed into a 6-vertex in $D(4, 6, 12)$. The 6-vertices in $D(4, 6, 12)$ have the property that they are in the interior of 2 edge-disjoint 6-cycles. The 3 adjacent 4-vertices, are in the interior of the outer of these 6-cycles, and vertices in the inner.

Due to these requirements, there are 5 possibilities for the 3 adjacent 4-vertices, giving 5 non-isomorphic possibilities of choosing these. Each of these 5 cases results in a contradiction, as indicated in Figure 14, by dotted arrows representing edges that do not exist in $M(3, 12^2)$.

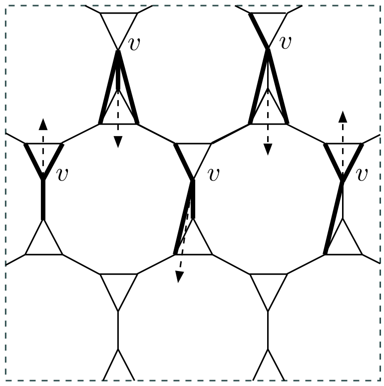

Figure 14: The dotted arrows show places where edges should exist if $D(4,6,12) \subseteq M(3,12^2)$

5.12 $D(4,8^2) \not\subseteq M(D(3,4,6,4))$

In $D(4,8^2)$ there exists a 4-cycle of 8-vertices, surrounding another vertex, which is adjacent to all 4 vertices in the cycle. Call this graph F.

There is only one way of finding F in $M(D(3,4,6,4))$, which however only is adjacent to 2 copies of F, and not 4, as in $D(4,8^2)$, see Figure 15.

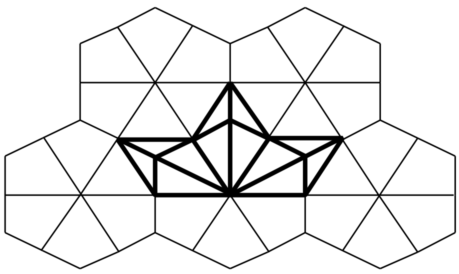

Figure 15: A copy of F with two adjacent copies.

5.13 $3^6 \not\subseteq M(3,6,3,6)$ and $M(3^4,6)$

In $M(3,6,3,6)$ and $M(3^4,6)$, there exists a cycle of six vertex-adjacent triangles, without a vertex in the interior. This does not exists in 3^6, and cannot be removed from $M(3,6,3,6)$ or $M(3^4,6)$ without creating polygons of size greater than 3.

5.14 $M(3^3, 4^2) \not\subseteq M(D(3,6,3,6))$

In $M(3^3, 4^2)$ all vertices have degree 7, and all triangles are adjacent to one K_4. In $M(D(3,6,3,6))$ there are vertices with degree 6, which cannot be removed, since then triangles that are not adjacent to any K_4's are created.

5.15 $D(4,6,12) \not\subseteq M(D(3^2,4,3,4))$, $M(D(3^3,4^2))$ and $M(3,6,3,6)$

Choose any vertex of degree 12 in $M(D(3^2,4,3,4))$, $M(D(3^3,4^2))$ or $M(3,6,3,6)$. In each case, there is a unique way of removing edges so that this vertex is incident to 12 triangles. These configurations cannot be extended to get $D(4,6,12)$.

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