Convexity and log-convexity of bond prices

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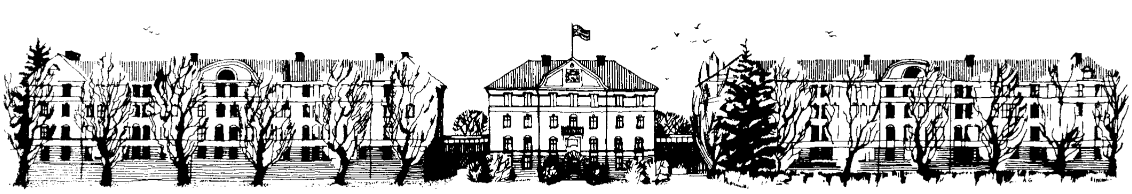

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Abstract

We consider convexity and monotonicity properties of bond prices and logarithms of bond prices. When explicit formulas are available they are frequently used, otherwise we rely on numerical methods.

Contents

Chapter 1 Introduction

Interest rate modeling theory was originally based on the assumption of specific one-dimensional dynamics for the instantaneous spot rate process r. The existence of a risk-neutral measure implies that the arbitrage-free price at time t of a contingent claim with payoff H_T at time T is given by

$$H_t = E_t\{D(t,T)H_T\} = E_t\{e^{-\int_t^T r(s)ds} H_T\}, \tag{1}$$

with E_t denoting the time t-conditional expectation under that measure . In particular, the zero-coupon-bond price at time t for the maturity T is characterized by a unit amount of currency available at time T, so that H_T =1 and we obtain

$$P(t,T) = E_t\{e^{-\int_t^T r(s)ds}\} \quad . \tag{2}$$

From (2) it is clear that whenever we can characterized the distribution of $e^{-\int_t^T r(s)ds}$ in terms of a chosen dynamics for r , conditional on the information available at time t, we are able to compute the bond prices P. As we all known, from bond prices all kind of rates are available, so that actually the whole zero-coupon curve is characterized in terms of distributional properties of r under the risk neutral measure.

In this paper we will mainly discuss the following short rate models: the Vasicek model, the Cox, Ingersoll, and Ross model, the Dothan model [1]. The bond prices $P(t,T) = E_t\{e^{-\int_t^T r(s)ds}\}$ of some models here are explicitly computable from dynamics, but some are found by using numerical methods. By analyzing bond prices of these models, we will discuss more about the price convexity as a function of interest rate and we will try to find out if the price is increasing in the volatility of these short rate models. We will also consider log-convexity of bond prices.

Chapter 2 Basic Mathematical and Financial terms

2.1 Short-rate models

In the context of interest rate derivatives, a short rate model is a mathematical model that describes the future evolution of interest rates by describing the future evolution of the short rate.

2.1.1 The short rate

The short rate, usually written r_t is the (annualized) interest rate at which an entity can borrow money for an infinitesimally short period of time from time t. Specifying the current short rate does not specify the entire yield curve. However no-arbitrage arguments show that, under some fairly relaxed technical conditions, if we model the evolution of r_t as a stochastic process under a risk-neutral measure Q then the price at time t of a zero-coupon bond maturing at time T is given by:

$$P(t,T) = E[\exp(-\int_t^T r_s ds) \mid F_t]$$

where F is the natural filtration for the process . Thus specifying a model for the short rate specifies future bond prices. This means that future instantaneous forward rates are also specified by the usual formula:

$$f(t,T) = -\frac{\partial}{\partial T}\ln(P(t,T)).$$

2.1.2 Short rate models

In this paper, we will discuss the following short rate models, here W_t represents a standard Brownian motion and dW_t its differential.

1. $dr_t = k[\theta - r_t]dt + \sigma\, dW_t$ (Vasicek model)
2. $dr_t = k[\theta - r_t]dt + \sigma\sqrt{r_t}\,dW_t$ (Cox, Ingersoll and Ross model)
3. $dr_t = ar_t dt + \sigma\, r_t dW_t$ (Dothan model)

2.2 Wiener process

A stochastic process $W(t)$ is a Wiener process (or Brownian motion) if:

1) $W(0) = 0$
2) W has continuous trajectories
3) W has independent increments (i.e. if $0 \leq t_1 < t_2 \leq t_3 < t_4$, then random variables $W(t_2) - W(t_1)$ and $W(t_4) - W(t_3)$ are independent.)
4) W has Gaussian increments: if $t_1 < t_2$, then

$$(W(t_2) - W(t_1)) \sim N(0, \sqrt{t_2 - t_1})$$

where $\sqrt{t_2 - t_1}$ denotes the standard deviation of $W(t_2) - W(t_1)$.

Remark: By an n-dimensional wiener process, we mean $W = (W_1, W_2, W_3, \ldots.., W_n)$ where each W_k is a wiener process and the components are mutually independent [2].

2.3 Ito's Lemma

In mathematics, Ito's lemma is used in stochastic calculus to find the differential of a function of a particular type of stochastic process. It is the stochastic calculus counterpart of the chain rule in ordinary calculus and is best memorized using the Taylor series expansion and retaining the second order term related to the stochastic component change. The lemma is widely employed in mathematical finance [3].

Suppose that the random process x is defined by the Ito process

$$dx(t) = a(x,t)dt + b(x,t)dz \qquad (*)$$

where z is a standard wiener process . Suppose also that process $y(t)$ is defined by $y(t) = F(x,t)$. Then $y(t)$ satisfies the Ito equation

$$dy(t) = (\frac{\partial F}{\partial x}a + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}b^2)dt + \frac{\partial F}{\partial x}bdz$$

where z is the same wiener process as in Equation (*).

2.4 Black-Scholes equation

The pioneering work of Black and Scholes [4] started the serious study of the theory of option pricing. All further advances in this field have been extensions and refinements of the original idea expressed in that paper.

Suppose that the arbitrary free market contains:

- An underlying security which its price governed by a geometric Brownian motion $dS(t) = S(t)\mu dt + S(t)\sigma dZ(t)$, process over a time interval $[0,T]$, where μ and σ are constants.
- A risk free asset with dynamics $dB(t) = rB(t)dt$, where the interest rate r is constant.
- A simple contingent claim of the form $\chi = \Phi(S(t))$ which can be traded on the market with the price process $\Pi(t)$.

Then the only pricing function of the form $\Pi(t) = F(t, S(t))$ which is consistent with the absence of arbitrage, is when F satisfies the following partial differential equation:

$$\frac{\partial F}{\partial t}(t,s) + rS\frac{\partial F}{\partial s}(t,s) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial s^2}(t,s) - rF(t,s) = 0 .$$

Subject to the boundary condition:

$$F(T,s) = \Phi(s)$$

in the strip $[0,T]\times\mathbb{R}_+$.

2.5 Convexity

In mathematics, a real-valued function f defined on an interval (or on any convex subset C of some vector space) is called convex, if for any two points x and y in its domain C and any t in [0,1], we have $f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$.

In other words, a function is convex if and only if its epigraph (the set of points lying on or above the graph) is a convex set.

A function is called strictly convex if

$$f(tx+(1-t)y) < tf(x)+(1-t)f(y)$$

for any t in (0,1) and $x \neq y$.

In finance, convexity is a measure of the sensitivity of the price of a bond to changes in interest rates. It is related to the concept of duration [5].

2.6 Log-convexity

A function $f(x)$ is logarithmically convex on the interval $[a,b]$ if $f > 0$ and $\ln f(x)$ is convex on $[a,b]$. We say that the price is log-convex if the logarithm of the price is convex in r. Convexity of a bond price means that the absolute value of the decline is decreasing in r, while log-convexity means that the relative decline diminishes when r grows [6].

2.7 Volatility

Volatility most frequently refers to the standard deviation of the change in value of a financial instrument with a specific time horizon. It is often used to quantify the risk of the instrument over that time period. Volatility is typically expressed in annualized terms, and it may either be an absolute number (*$5*) or a fraction of the initial value (*5%*).

For a financial instrument whose price follows a Gaussian random walk, or Wiener process, the volatility increases by the square-root of time as time increases. Conceptually, this is because there is an increasing probability that the instrument's price will be farther away from the initial price as time increases.

More broadly, volatility refers to the degree of (typically short-term) unpredictable change over time of a certain variable. It may be measured via the standard deviation of a sample, as mentioned above. However, price changes actually do not follow Gaussian distributions. Better distributions used to describe them actually have "fat tails" although their variance remains finite. Therefore, other metrics may be used to describe the degree of spread of the variable. As such, volatility reflects the degree of risk faced by someone with exposure to that variable.

Historical volatility is the volatility of a financial instrument based on historical returns. This phrase is used particularly when it is wished to distinguish between the actual volatility of an instrument in the past, and the current volatility implied by the market [7].

Chapter 3 Numerical methods

3.1 Introduction

Numerical analysis involves the study of methods of computing numerical data. In many problems this implies producing a sequence of approximations; thus the questions involve the rate of convergence, the accuracy (or even validity) of the answer, and the completeness of the response. (With many problems it is difficult to decide from a program's termination whether other solutions exist.) Since many problems across mathematics can be reduced to linear algebra, this too is studied numerically; here there are significant problems with the amount of time necessary to process the initial data. Numerical solutions to differential equations require the determination not of a few numbers but of an entire function; in particular, convergence must be judged by some global criterion. Other topics include numerical simulation, optimization, and graphical analysis, and the development of robust working code.

For solving partial differential equations, numerical analysis method is one of the mostly common used methods. There are many different numerical analysis methods, such as: finite difference method, finite element method, finite volume method etc. Here, in my paper, we will mainly discuss the finite difference method.

3.2 Finite difference method

The finite difference method (FDM) was first developed by A. Thom [8] in the 1920s under the title “the method of square” to solve nonlinear hydrodynamic equations.

The finite difference techniques are based upon the approximations that permit replacing differential equations by finite difference equations. These finite difference approximations are algebraic in form, and the solutions are related to grid points.

Thus, a finite difference solution basically involves three steps:

1. Dividing the solution into grids of nodes.
2. Approximating the given differential equation by finite difference equivalence that relates the solutions to grid points.
3. Solving the difference equations subject to the prescribed boundary conditions and (or) initial conditions.

Given a function *f(x)* shown in Figure1, we can approximate its derivative, slope or tangent at **P** by the slope of the arcs **PB**, **PA**, or **AB**, for obtaining the forward difference, backward-difference, and central-difference formulas respectively.

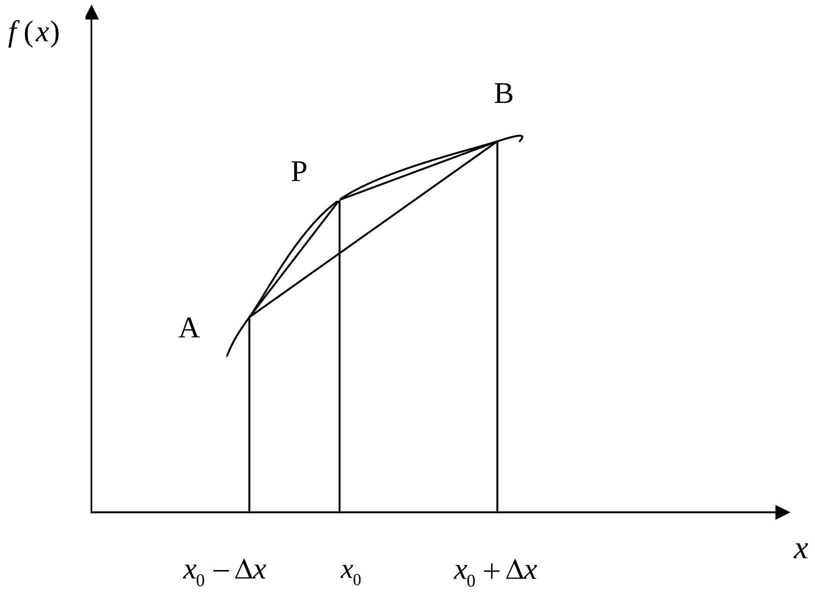

Figure 1 Estimates for the derivative of $f(x)$ at P by using forward, backward and central differences

Here is the table of formulas for finite difference method.

forward-difference formula

$$f'(x_0)\cong\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

backward-difference formula

$$f'(x_0)\cong\frac{f(x_0)-f(x_0-\Delta x)}{\Delta x}$$

central-difference formula

$$f'(x_0)\cong\frac{f(x_0+\Delta x)-f(x_0-\Delta x)}{2\Delta x}$$

The approach used for obtaining above finite difference equations is Taylor's series:

$$f(x_0+\Delta x)=f(x_0)+\Delta x f'(x_0)+\frac{1}{2!}(\Delta x)^2 f''(x_0)+\frac{1}{3!}(\Delta x)^3 f'''(x_0)+O(\Delta x)^4 \qquad (3)$$

and

$$f(x_0-\Delta x)=f(x_0)-\Delta x f'(x_0)+\frac{1}{2!}(\Delta x)^2 f''(x_0)-\frac{1}{3!}(\Delta x)^3 f'''(x_0)+O(\Delta x)^4 \qquad (4)$$

where $O(\Delta x)^4$ is the error introduced by truncating the series.

If we subtract (3) by (4), then we get

$$f(x_0+\Delta x)-f(x_0-\Delta x)=2\Delta x f'(x_0)+O(\Delta x)^3.$$

We can re-write it as

$$f'(x_0)\cong\frac{f(x_0+\Delta x)-f(x_0-\Delta x)}{2\Delta x}+O(\Delta x)^2,$$ and this is

central-difference formula. Note that the $O(\Delta x)^2$ means the truncation error is the order of $(\Delta x)^2$ for the central difference.

The forward-difference and backward-difference formulas could be obtained by re-arranging (3) and (4) respectively, and we have

$$f'(x_0)\cong\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}+O(\Delta x),$$ (for forward difference)

and

$$f'(x_0)\cong\frac{f(x_0)-f(x_0-\Delta x)}{\Delta x}+O(\Delta x)$$ (for backward difference). We can find the truncation errors of these two formulas are of order Δx.

By adding (3) and (4),

$$f(x_0+\Delta x)-f(x_0-\Delta x)=2f(x_0)+(\Delta x)^2 f''(x_0)+O(\Delta x)^4,$$

and we have

$$f''(x_0)\cong\frac{f(x_0+\Delta x)-2f(x_0)+f(x_0-\Delta x)}{(\Delta x)^2}+O(\Delta x)^2.$$

Higher order finite difference approximations can be obtained by taking more terms in Taylor series expansion.

To apply the difference method to find the solution of a function $F(x,t)$, we divide the solution region in $x-t$ plane into equal rectangles or meshes of sides Δx and Δt. The derivatives of F at the $(i,j)^{th}$ node are shown in the table (Table 1), where

$$\begin{aligned} x&=i\bullet\Delta x\\ t&=j\bullet\Delta t \end{aligned}.$$

$$F_{x|i,j} \simeq \frac{F(i+1,j)-F(i-1,j)}{2\Delta x},$$
$$F_{t|i,j} \simeq \frac{F(i,j+1)-F(i,j)}{\Delta t},$$
$$F_{xx|i,j} \simeq \frac{F(i+1,j)-2F(i,j)+F(i-1,j)}{(\Delta x)^2},$$
$$F_{tt|i,j} \simeq \frac{F(i,j+1)-2F(i,j)+F(i,j-1)}{(\Delta t)^2}$$

Table 1

The values of F along the first time row (see Figure2), $t = \Delta t$,can be calculated in terms of the boundary and initial conditions, then the values of F along the second row, $t = 2\Delta t$,are calculated in terms of the first time row, and so on.

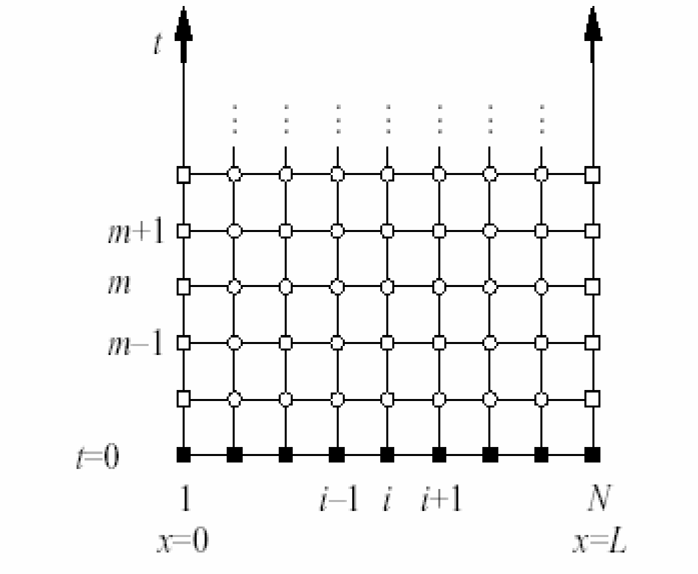

Figure 2 Finite difference mesh for two independent variable x and t

Reducing the mesh size could increase accuracy, but the mesh size could not be infinitesimal.

Decreasing the truncation error by using a finer mesh may result in increasing the round-off error due to the increased number of arithmetic operations. A point is reached where minimum total error occurs for any particular algorithm using any given word length.

If the partial differential equation has the continuous derivative on time space, we probably can discrete on time space, then Euler forward, Euler backward and Crank-Nicholson method can be chosen. Here Table 2 shows these three different methods.

Euler forward	$\frac{F^n - F^{n-1}}{\Delta t} + AF^n = 0$
Euler backward	$\frac{F^n - F^{n-1}}{\Delta t} + AF^{n-1} = 0$
Crank-Nicholson	$\frac{F^n - F^{n-1}}{\Delta t} + \frac{1}{2} A(F^n + F^{n-1}) = 0$

Table 2

In Table 2, AF^n denotes the equations after the finite difference substitution in domain.

When using finite difference method, it is very important to apply the boundary conditions. It represents the value or the function lying on the boundary of the domain. Table 3 shows three kinds of boundary condition:

Dirichlet boundary condition	$F(\partial\Omega) = u$
Neumann boundary condition	$\frac{\partial F(\partial\Omega)}{\partial n} = u$
Cauchy boundary condition	$aF(\partial\Omega) + b\frac{\partial F(\partial\Omega)}{\partial n} = u$

Table 3 boundary conditions

In the table $\partial\Omega$ denotes the boundary of domain, and u is a deterministic function or value, a and b are some certain numbers.

Chapter 4 One-factor short-rate models

4.1 The Cox, Ingersoll and Ross (CIR) model

The Cox, Ingersoll and Roll (CIR) [9] model assumes that short rate $r(t)$ satisfies

$$dr(t) = k(\theta - r(t))dt + \sigma\sqrt{r(t)}dW(t), r(0) = r_0 > 0 \tag{4.1}$$

where r_0, k, θ, σ are positive constants. The CIR model has been a benchmark for many years because of its analytical tractability and the fact that, contrary to the Vasecik [10] model, the instantaneous short rate is always positive. The condition $2k\theta > \sigma^2$ has to be imposed to ensure that the origin is inaccessible to the process (4.1), so that we can grant that r remains positive.

The price at time t of a zero-coupon bond with maturity T is

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)} \tag{4.2}$$

Where

$$A(t,T) = [\frac{2h\exp\{(k+h)(T-t)/2\}}{2h+(k+h)(\exp\{(T-t)h\}-1)}]^{2k\theta/\sigma^2}$$

$$B(t,T) = \frac{2(\exp\{(T-t)h\}-1)}{2h+(k+h)(\exp\{(T-t)h\}-1)}$$

$$h = \sqrt{k^2 + 2\sigma^2}\ .$$

From (4.2), we can easily to find out if the bond price is convex as a function of r or not by taking the second derivative of the price function with respect to the interest rate $r(t)$.

We have:

$$\frac{\partial}{\partial r}P(t,T) = A(t,T)(-B(t,T))e^{-B(t,T)r(t)},$$

$$\frac{\partial^2}{\partial r^2}P(t,T) = A(t,T)B(t,T)^2 e^{-B(t,T)r(t)}\ .$$

It is obvious that $\frac{\partial^2}{\partial r^2}P(t,T) > 0$, since $A(t,T) > 0$. By taking the second derivative with respect to $r(t)$, we know that the bond price is convex as a function of interest rate r for CIR model.

Now we want to find out if the bond price is increasing in the volatility of CIR model or not. It seems not very easy to take derivatives of σ. For CIR model, the model formulation under the risk-neutral measure Q is $dr(t) = k(\theta - r(t))dt + \sigma\sqrt{r(t)}dW(t)$. Given this assumption, a pure discount bond price is a function of the current interest rate r, the current time t, and the maturity time T of the bond. Under arbitrage free conditions, the value of an interest rate contingent claim $B(t,T,r)$ satisfies the following PDE for CIR model

$$\begin{aligned}&\frac{\partial B}{\partial t} + (k\theta - kr)\frac{\partial B}{\partial r} + \frac{1}{2}\sigma^2 r\frac{\partial^2 B}{\partial r^2} - rB = 0 \\ &B(T,T,r) = 1\end{aligned}. \tag{4.3}$$

By solving the PDE above, we can get the bond price. Now next we will use numerical methods to solve our equation.

4.1.1 Numerical solution of CIR model

We already discussed the PDE of CIR model in last section, now we will solve the equation (4.3). Since the order of accuracy is 2, we will use Crank-Nicholson to solve this partial differential equation. Euler backward has the order of accuracy 1, and Euler forward depends on strictly stable conditions. It is obvious that in (4.3), there are terms of second order derivative. We will use the following expressions to approximate:

$$\frac{\partial^2 F}{\partial x^2} = \frac{F(x+\Delta x) - 2F(x) + F(x-\Delta x)}{(\Delta x)^2}. \tag{4.4}$$

4.1.1.1 Boundary condition

Since CIR model is a “square-root” term in the diffusion coefficient of the instantaneous short-rate dynamics [11], the domain for this model is $r = [0, \infty)$.

As $r \to \infty$, $B(t, r \to \infty) = 0$. In (4.2), if we let $r = 0$, then we will get:

$P(t,T,0) = A(t,T)$, this is exactly the boundary condition for $r = 0$.

4.1.1.2 Iterative process

We will use backward difference to approximate the first derivative with respect to time t, but we choose central difference to approximate the first derivative with respect to interest rate r. Since we already discuss before, formula (4.4) is chosen for approximating the second derivative with respect to interest rate r.

Crank-Nicholson scheme is applied for solving the differential equation.
Since

$$\frac{\partial B}{\partial t} = \frac{F_j^n - F_j^{n-1}}{dt},$$

$$\frac{\partial B}{\partial r} = \frac{F_{j+1}^n - F_{j-1}^n}{2h},$$

$$\frac{\partial^2 B}{\partial r^2} = \frac{F_{j+1}^n - 2F_j^n + F_{j-1}^n}{h^2}$$

Where $\begin{aligned} dt &= \Delta t \\ h &= \Delta h \end{aligned}$

We can also approximate r with this $r = i\Delta h = ih$. ($j = 1.......N$)

If we plug them into equation (4.3), then we obtain:

$$(k\theta - kr)\frac{\partial B}{\partial r} + \frac{1}{2}\sigma^2 r \frac{\partial^2 B}{\partial r^2} - rB$$

$$= (k\theta - kih)\frac{F_{j+1}^n - F_{j-1}^n}{2h} + \frac{1}{2}\sigma^2 ih \frac{F_{j+1}^n - 2F_j^n + F_{j-1}^n}{h^2} - ihF_j^n$$

$$= \frac{(k\theta - kih + \sigma^2 i)}{2h} F_{j+1}^n + (-(ih + \frac{i\sigma^2}{h}))F_j^n + \frac{(i\sigma^2 - k\theta + kih)}{2h} F_{j-1}^n$$

$$= AF^n + b. \qquad (j = 1.......N)$$

where

$$A = \begin{pmatrix} -(ih + \frac{i\sigma^2}{h}) & \frac{i\sigma^2 + k\theta - kih}{2h} & 0 \\ \frac{i\sigma^2 - k\theta + kih}{2h} & \ddots & \frac{i\sigma^2 + k\theta - kih}{2h} \\ 0 & (b_{N-1} - c_{N+1}) & 2c_{N+1} + a_N \end{pmatrix}_{N \times N}$$

$$b_{N-1} = \frac{i\sigma^2 - k\theta + kih}{2h}|_{i=N-1}$$

$$a_N = (-(ih + \frac{i\sigma^2}{h}))_{i=N}$$

$$c_{N+1} = \frac{i\sigma^2 + k\theta - kih}{2h}|_{i=N+1}$$

$$b = \begin{pmatrix} (\sigma^2 - k\theta + kh)\frac{1}{2h} * (A(n\Delta t, T)) \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ \end{pmatrix}_{N\times 1} .$$

We use Crank-Nicholson scheme to get the approximate solution, and now the equation will be changed as:

$$\frac{F^n - F^{n-1}}{dt} + \frac{1}{2}A(F^n + F^{n-1}) + \frac{1}{2}(b^n + b^{n-1}) = 0$$

$$\Rightarrow (\frac{1}{2}A - \frac{I}{dt})F^{n-1} = -(\frac{1}{2}A + \frac{I}{dt})F^n - \frac{1}{2}(b^n + b^{n-1})$$

$$\Rightarrow A_1 F^{n-1} = B_1 F^n + b_1 .$$

$$A_1 = (\frac{1}{2}A - \frac{I}{dt})$$

$$B_1 = -(\frac{1}{2}A + \frac{I}{dt})$$

$$b_1 = -\frac{1}{2}(b^n + b^{n-1})$$

Where A_1, B_1 are both $N \times N$ matrix, and I is $N \times N$ identity matrix, while N is the number of how many steps of the interest rate is divided.

4.1.1.3 Implementation and results

Here we will present the numerical solution graphically. For CIR model, it is very important for us to choose the appropriate parameter.

Parameters:

$\sigma = 0.3$
$\theta = 0.08$
$k = 0.12$.
$r_{\max} = 0.5$
$T = 3$

We have the numerical solution as shown in Figure3:

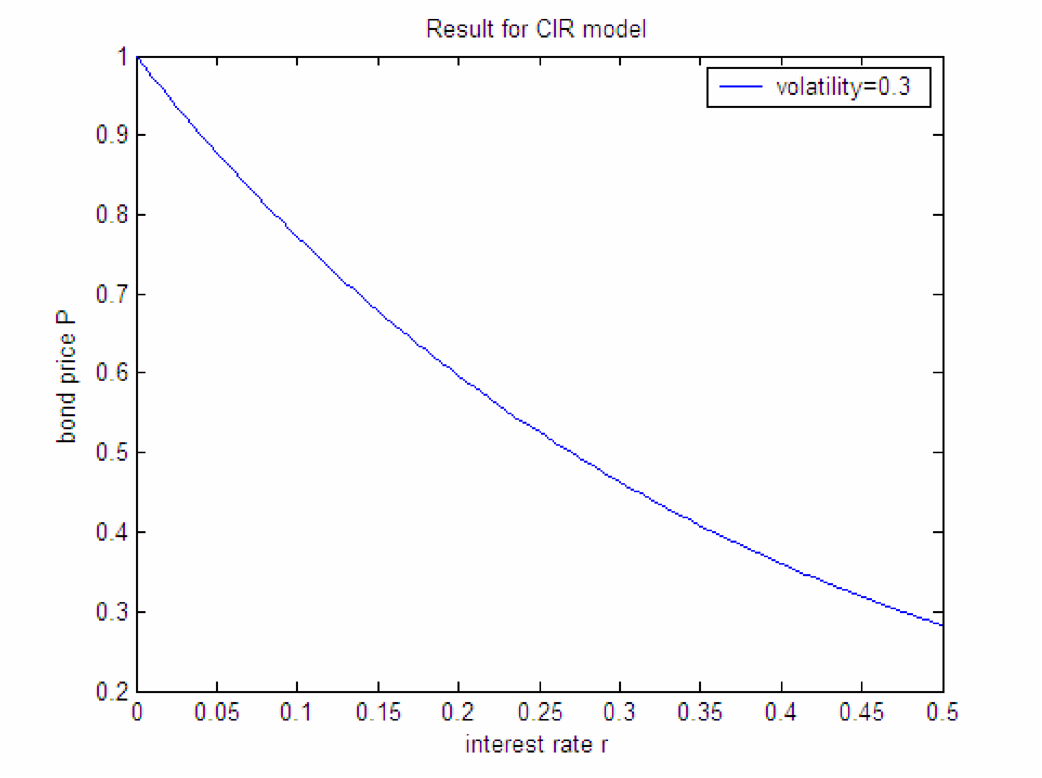

Figure 3 Result from finite difference methods for CIR model

From Figure3, we can see the result is pretty nice, and we can easily see from the graph that bond price is convex as a function of r .

Now let us discuss the convexity of the logarithm of bond price for CIR model. If we take logarithm on both sides of (4.2), we obtain:

$$\log P(t,T) = (-B(t,T)r(t))\log(A(t,T)) . \tag{4.5}$$

It is easy to find out that logarithm of bond price is a linear function of interest rate $r(t)$. This can also be testified by our numerical solution, and we will use the same parameters.

For Figure3, we get the corresponding graphs as shown in Figure4.

Parameters:

$\sigma = 0.3$
$\theta = 0.08$
$k = 0.12$.
$r_{\max} = 0.5$
$T = 3$

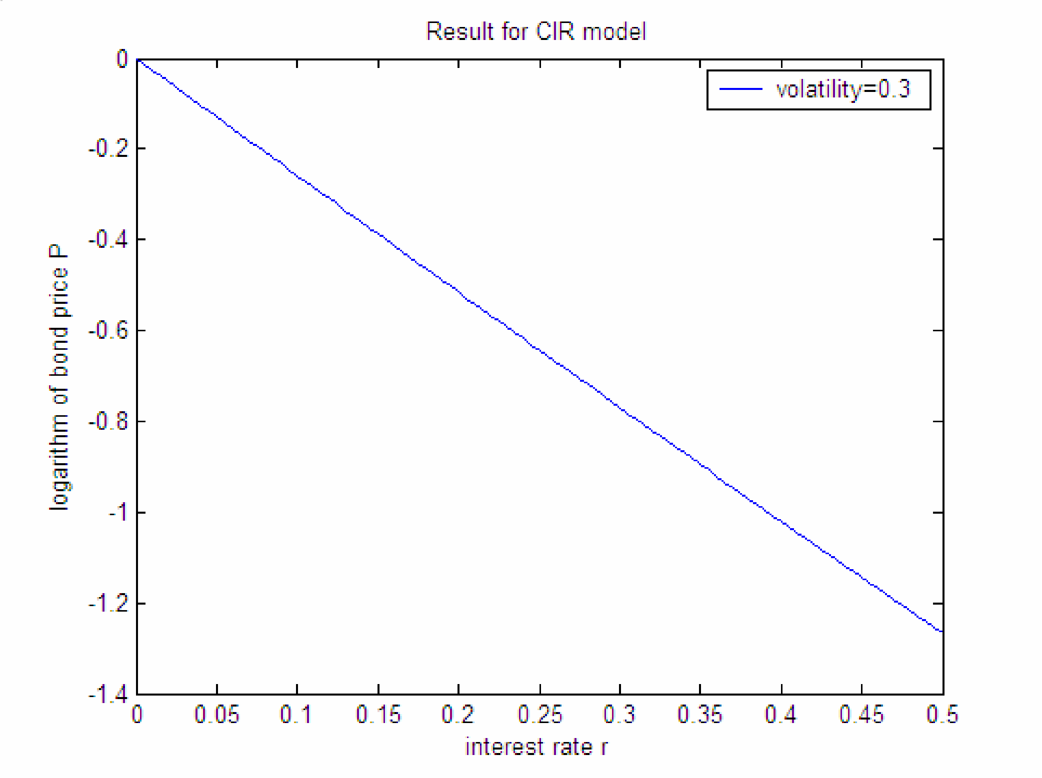

Figure 4 Logarithm of bond price for CIR model

From Figure4, we can see that the logarithm of bond price is also convex as a function of interest rate r for CIR model.

From the numerical solutions, we can also check if the bond price increases with the volatility in the CIR model or not. Here we choose different volatility for three groups of parameters, where we keep all the other parameters the same value except volatility.

Parameters:

$\theta = 0.08$
$k = 0.12$
$r_{\max} = 0.5$
$T = 3$

$\sigma_1 = 0.3$
$\sigma_2 = 0.4$.
$\sigma_3 = 0.5$

We get the following graph:

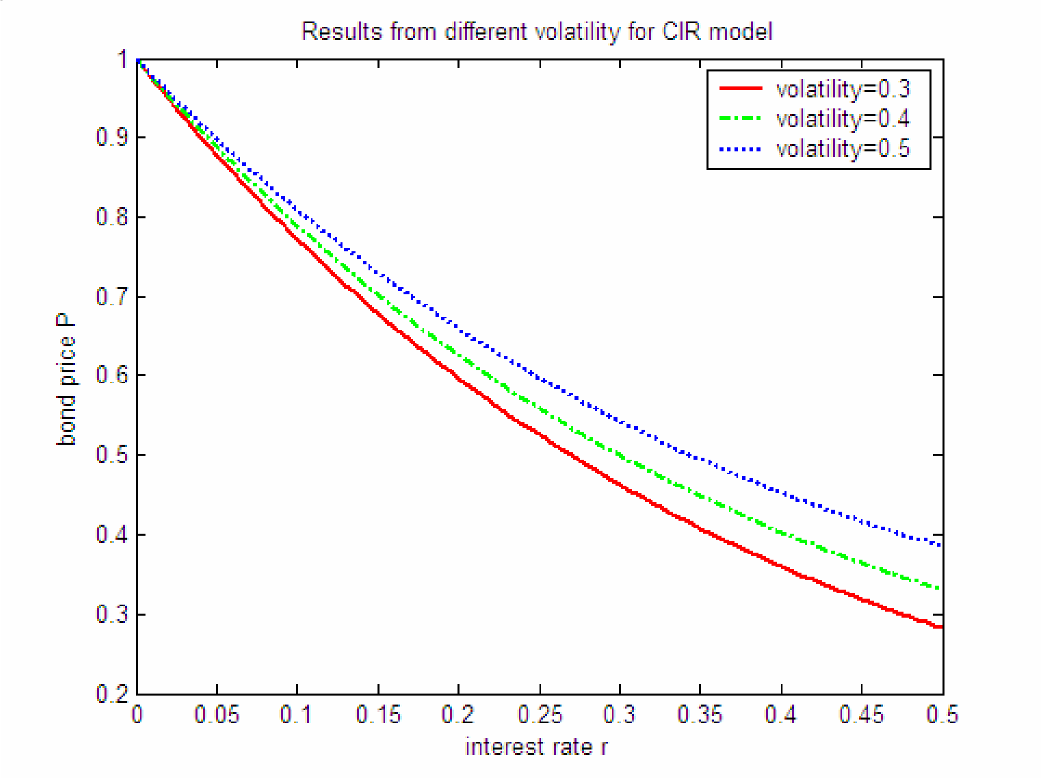

Figure 5 Results with different volatility for CIR model

From Figure5, it is very easy to find out that the bond prices increase in the volatility of CIR model.

Also under the same parameters as Figure5, we get the graph for logarithm of bond price for different values of volatility as shown in Figure6.

Parameters:

$\theta = 0.08$
$k = 0.12$
$r_{\max} = 0.5$
$T = 3$

$\sigma_1 = 0.3$
$\sigma_2 = 0.4$.
$\sigma_3 = 0.5$

The graph:

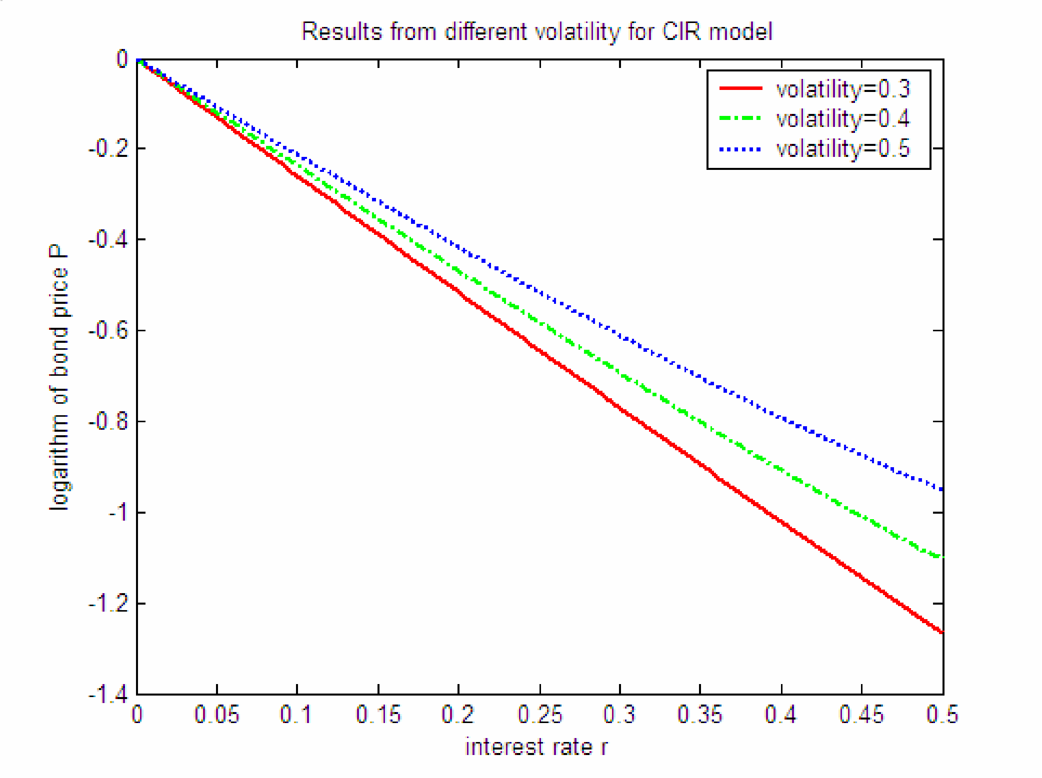

Figure 6 Logarithm of bond price with different volatility for CIR model

From the Figure6, we can conclude that logarithm of bond prices increase in the volatility of CIR model.

4.2 The Vasicek model

Vasicek assumed that the short rate r_t follows a Gaussian model

$$dr(t) = k[\theta - r(t)]dt + \sigma\, dW(t) \quad , \quad r(0) = r_0 \tag{4.6}$$

where as usual $W(t)$ is a Q-Brownian motion, r_0, k and σ are positive constants.

Integrating equation (4.6), we get, for each $s \le t$,

$$r(t) = r(s)e^{-k(t-s)} + \theta(1-e^{-k(t-s)}) + \sigma\int_s^t e^{-k(t-u)}dW(u) \quad . \tag{4.7}$$

As we known, the price of a pure-discount bond can be derived by computing the expectation of $P(t,T) = E_t\{e^{-\int_t^T r(s)ds}\}$. We look for a solution of the form:

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)} \tag{4.8}$$

and find that:

$$A(t,T) = \exp\{(\theta - \frac{\sigma^2}{2k^2})[B(t,T) - T + t] - \frac{\sigma^2}{4k}B(t,T)^2\}$$

$$B(t,T) = \frac{1}{k}[1 - e^{-k(T-t)}] .$$

From (4.8), we can easily to find out if the bond price is convex as a function of r or not by taking the second derivative of the price function with respect to the interest rate $r(t)$.

We have

$$\frac{\partial}{\partial r}P(t,T) = A(t,T)(-B(t,T))e^{-B(t,T)r(t)} ,$$

$$\frac{\partial^2}{\partial r^2}P(t,T) = A(t,T)B(t,T)^2e^{-B(t,T)r(t)} \quad .$$

It is obvious that $\frac{\partial^2}{\partial r^2}P(t,T)$ >0, since $A(t,T)$ >0.

By taking the second derivative with respect to $r(t)$, we know that the bond price is convex as a function of r .

Now our next step is to discuss the volatility of (4.8), and that is the σ in the formula.

From the discussion above, we know that the bond price is convex as a function of r . If check (4.8), we will notice that σ^2 also have some impact on the bond price, and our next step is to find out what kind of relationship between σ^2 and bond price $P(t,T)$. From (4.8), we notice in the formula bond price as a function of σ^2 , and also it is easy to take σ^2 as a variable instead of σ . Here in order to simplify the mathematical deduction, we replace σ^2 as another variable z, that is $\sigma^2 = z$. Then we have:

$$\frac{\partial P}{\partial z} = A(t,T)e^{-B(t,T)r(t)}\{-\frac{1}{2k^2}[B(t,T)-T+t]-\frac{1}{4k}B(t,T)^2\}.$$

It is easy to find out that $A(t,T)e^{-B(t,T)r(t)} > 0$, what we need prove now is that the

$$-\frac{1}{2k^2}[B(t,T)-T+t]-\frac{1}{4k}B(t,T)^2 \geq 0. \tag{4.9}$$

By substituting $B(t,T)=\frac{1}{k}[1-e^{-k(T-t)}]$ into (4.9), we get

$$-\frac{1}{2k^2}[\frac{1}{k}(1-e^{-k(T-t)})-T+t]-\frac{1}{4k}\frac{1}{k^2}[1-e^{-k(T-t)}]^2$$
$$=-\frac{1}{4k^3}[2-2e^{-k(T-t)}]-\frac{1}{4k^3}[1-2e^{-k(T-t)}+e^{-2k(T-t)}]+\frac{1}{2k^2}(T-t)$$
$$=-\frac{1}{4k^3}[3-4e^{-k(T-t)}+e^{-2k(T-t)}]+\frac{2}{4k^3}k(T-t)$$
$$=\frac{1}{4k^3}[2k(T-t)-3+4e^{-k(T-t)}-e^{-2k(T-t)}]. \tag{4.10}$$

Since $k > 0$ and $T-t \geq 0$, that means $k(T-t) \geq 0$, let $k(T-t) = x$, we get a function

$f(x) = 2x-3+4e^{-x}-e^{-2x}$ ($x \geq 0$).

It's easy to know that $f'(x) = 2-4e^{-x}+2e^{-2x} = 2(1-e^{-x})^2 > 0$, also $f'(x)=0$ if $x=0$. That means $f(x)$ is increasing if $x \geq 0$, and also it is easy to find out if $x=0$, we will get the minimal value $f(0)=0$. Now we get back to (4.10), from our deduction, we know that $\frac{1}{4k^3}[2k(T-t)-3+4e^{-k(T-t)}-e^{-2k(T-t)}]$ >0 if $T > t$, and $\frac{1}{4k^3}[2k(T-t)-3+4e^{-k(T-t)}-e^{-2k(T-t)}]$=0 if $T=t$. By doing this , we proved that (4.9) ≥ 0 , that means $\frac{\partial P}{\partial z} \geq 0$, since the zero value can only get if $T=t$, that also means our bond price is increasing if σ^2 is increasing, since in the Vasicek model σ is positive constant, then we can say bond price is increasing in the volatility of Vasicek model.

Now let us discuss the convexity of the logarithm of bond price. If we take logarithm on both sides of (4.8), we obtain:

$$\log P(t,T) = (-B(t,T)r(t))\log(A(t,T)) . \qquad (4.11)$$

If we substitute the value of $A(t,T)$ and $B(t,T)$ into (4.11), we get:

$$\log P(t,T) = r(t)\{(\theta - \frac{\sigma^2}{2k^2})(\frac{1}{k}[1-e^{-k(T-t)}]-T+t)-\frac{\sigma^2}{4k}\frac{1}{k^2}[1-e^{-k(T-t)}]^2\}(-\frac{1}{k}[1-e^{-k(T-t)}]).$$

It’s obvious that the logarithm of bond price is a linear function of interest rate r, so it’s convex as a function of r.

We can also use numerical method to show convexity and monotonicity properties of bond price graphically.

Under arbitrage free conditions, the value of an interest rate contingent claim $B(t,T,r)$ satisfies the following PDE for Vasicek model

$$\begin{aligned} &\frac{\partial B}{\partial t} + (k\theta - kr)\frac{\partial B}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 B}{\partial r^2} - rB = 0 \\ &B(T,T,r) = 1 \end{aligned} . \qquad (4.12)$$

If $r = 0$, from (4.8), we have $P(t,T,0) = A(t,T)$, this is the boundary condition.

By solving the PDE above, we can get the bond price.

We will use the same method to solve (4.12) as we did for (4.3). Since we already discussed in the last section how to solve the PDE for CIR model, here we will simplify the procedure. For Vasicek model, we have:

$$\begin{aligned} &(k\theta - kr)\frac{\partial B}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 B}{\partial r^2} - rB \\ &= (k\theta - kih)\frac{F_{j+1}^n - F_{j-1}^n}{2h} + \frac{1}{2}\sigma^2\frac{F_{j+1}^n - 2F_j^n + F_{j-1}^n}{h^2} - ihF_j^n \\ &= (\frac{\sigma^2}{2h^2} + \frac{k\theta - kih}{2h})F_{j+1}^n + (-(ih + \frac{\sigma^2}{h^2}))F_j^n + (\frac{\sigma^2}{2h^2} - \frac{k\theta - kih}{2h})F_{j-1}^n \\ &= AF^n + b . \qquad (\, j = 1.......N \,) \end{aligned}$$

where

$$A=\begin{pmatrix} -(ih+\frac{\sigma^2}{h^2}) & \frac{\sigma^2}{2h^2}+\frac{k\theta-kih}{2h} & 0 \\ \frac{\sigma^2}{2h^2}-\frac{k\theta-kih}{2h} & \ddots & \frac{\sigma^2}{2h^2}+\frac{k\theta-kih}{2h} \\ 0 & (b_{N-1}-c_{N+1}) & 2c_{N+1}+a_N \end{pmatrix}_{N\times N}$$

$$b_{N-1}=(\frac{\sigma^2}{2h^2}-\frac{k\theta-kih}{2h})\,|_{i=N-1}$$

$$a_N=-(ih+\frac{\sigma^2}{h^2})\,|_{i=N}$$

$$c_{N+1}=(\frac{\sigma^2}{2h^2}+\frac{k\theta-kih}{2h})\,|_{i=N+1}$$

$$b=\begin{pmatrix} (\frac{\sigma^2}{2h^2}-\frac{k\theta-kh}{2h})*(A(n\Delta t,T)) \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix}_{N\times 1} .$$

We use Crank-Nicholson scheme to get the approximate solution, and now the equation will be changed as:

$$\frac{F^n-F^{n-1}}{dt}+\frac{1}{2}A(F^n+F^{n-1})+\frac{1}{2}(b^n+b^{n-1})=0$$

$$\Rightarrow (\frac{1}{2}A-\frac{I}{dt})F^{n-1}=-(\frac{1}{2}A+\frac{I}{dt})F^n-\frac{1}{2}(b^n+b^{n-1})$$

$$\Rightarrow A_1F^{n-1}=B_1F^n+b_1 .$$

$$A_1=(\frac{1}{2}A-\frac{I}{dt})$$

$$B_1=-(\frac{1}{2}A+\frac{I}{dt})$$

$$b_1=-\frac{1}{2}(b^n+b^{n-1})$$

Where A_1, B_1 are both $N\times N$ matrix, and I is $N\times N$ identity matrix, while N is the number of how many steps of the interest rate is divided.

We use Matlab to implement our numerical solution, and we get the following results.

Parameters:

$\sigma = 0.2$
$k = 0.12$
$\theta = 0.1$
$r_{\max} = 0.35$
$T = 3$

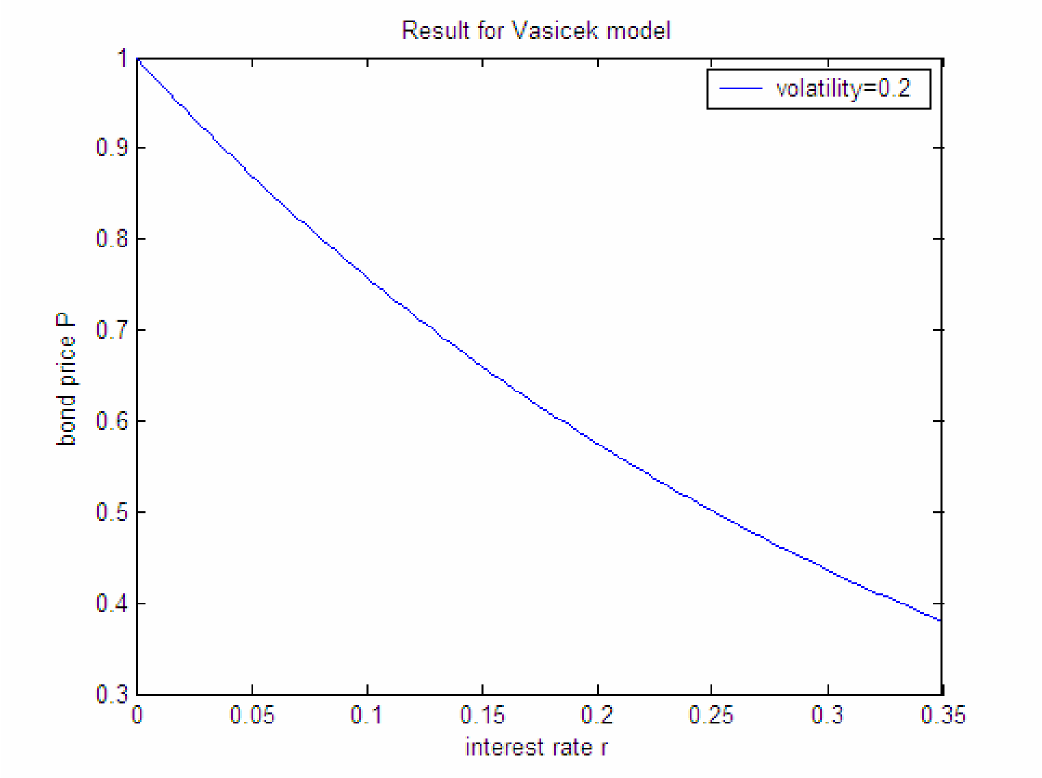

Figure 7 Result from finite difference methods for Vasicek model

From Figure7, we can see from the graph that bond price is convex as a function of interest rate r.

If we choose the same parameters, we will get the corresponding graph for logarithm of bond price.

Parameters:

$\sigma = 0.2$
$k = 0.12$
$\theta = 0.1$
$r_{\max} = 0.35$
$T = 3$

Graph:

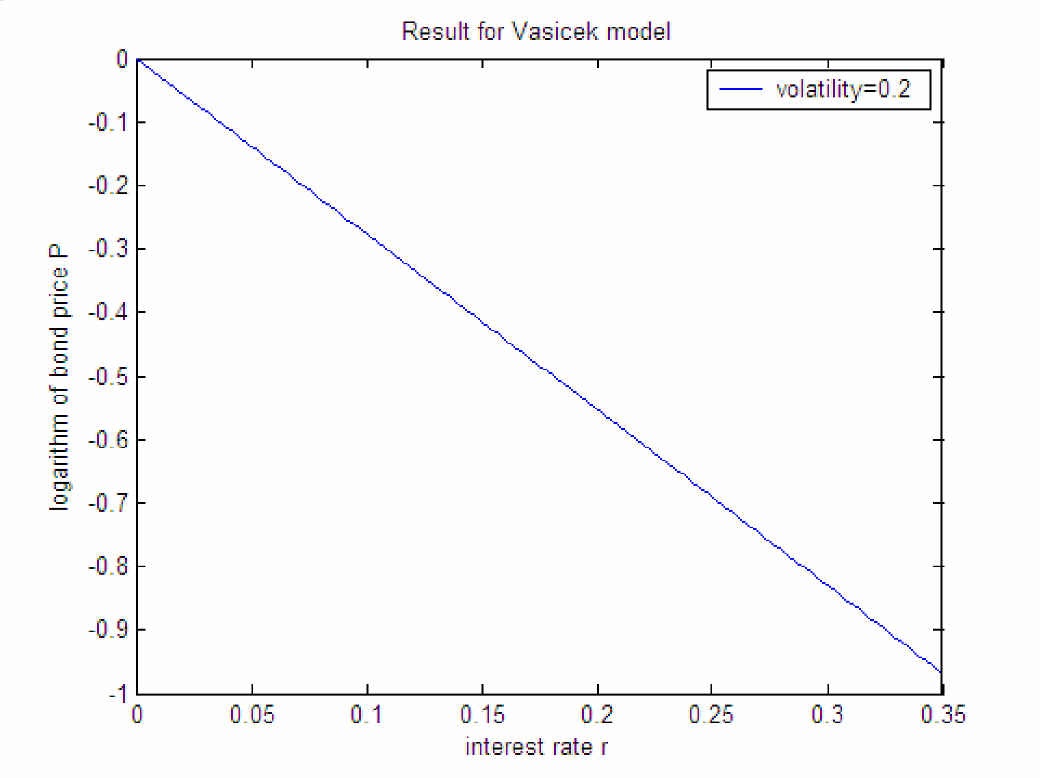

Figure 8 Logarithm of bond price for Vasicek model

From Figure8, we can see that the logarithm of bond price is also convex as a function of interest rate r.

From the numerical solutions, we can also check if the bond price increases with the volatility in the Vasicek short rate model or not. Here we choose different volatility for three groups of parameters, where we keep all the other parameters the same value except volatility.

Parameters:

$k = 0.12$
$\theta = 0.1$
$r_{\max} = 0.35$
$T = 3$

$\sigma_1 = 0.25$
$\sigma_2 = 0.3$,
$\sigma_3 = 0.35$

and we get:

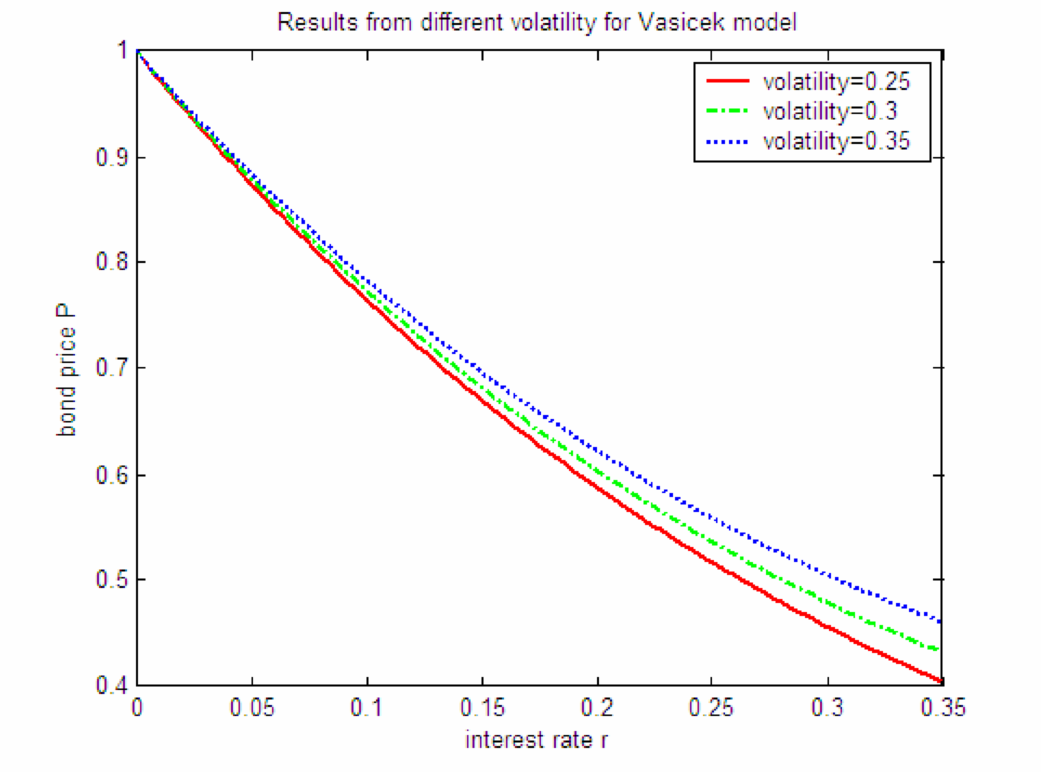

Figure 9 Results from different volatility for Vasicek model

From Figure9, it is very easy to find out that the bond prices increase in the volatility of Vasicek model.

If we keep all the parameters the same, we will get the corresponding graph for logarithm of bond prices for Vasicek model.

Parameters:

$k = 0.12$
$\theta = 0.1$
$r_{\max} = 0.35$
$T = 3$

$\sigma_1 = 0.25$
$\sigma_2 = 0.3$
$\sigma_3 = 0.35$

We get:

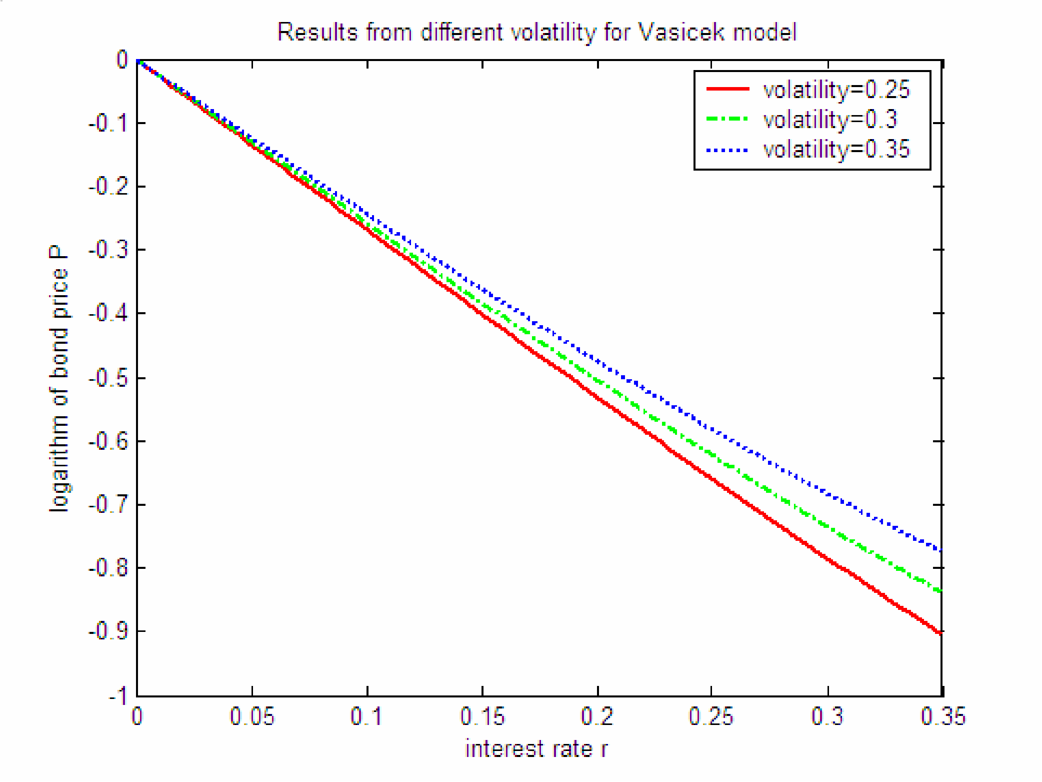

Figure 10 Logarithm of bond price with different volatility for Vasicek model

From Fihure10, we can see that logarithm of the bond prices also increase in the volatility of Vasicek model.

4.3 The Dothan model

Dothan [12] introduced a constant market price of risk, which is equivalent to directly assuming a risk-neutral dynamics of type

$$dr_t = ar_t dt + \sigma r_t dW_t$$

Where a is a real constant, σ is a positive constant.

The zero-coupon bond price derived by Dothan is given by

$$P(t,T)=\frac{\bar{r}^{p}}{\pi^{2}}\int_{0}^{\infty}\sin(2\sqrt{\bar{r}}\sinh y)\int_{0}^{\infty}f(z)\sin(yz)dzdy+\frac{2}{\Gamma(2p)}\bar{r}^{p}K_{2p}(2\sqrt{\bar{r}}) \qquad (4.13)$$

where

$$f(z)=\exp[\frac{-\sigma^{2}(4p^{2}+z^{2})(T-t)}{8}]z\,|\,\Gamma(-p+i\frac{z}{2})|^{2}\cosh\frac{\pi z}{2},$$
$$\bar{r}=\frac{2r(t)}{\sigma^{2}},$$
$$p=\frac{1}{2}-a,$$

and K_q denotes the modified Bessel function of the second kind of order q.

Though somehow explicit, formula (4.13) is rather complex since it depends on two integrals of functions involving hyperbolic sines and cosines. A double numerical integration is needed so that the advantage of having an “explicit” formula is dramatically reduced.

Under arbitrage free conditions, the value of an interest rate contingent claim $B(t,T,r)$ satisfies the following PDE for Dothan model

$$\begin{aligned}&\frac{\partial B}{\partial t}+ar\frac{\partial B}{\partial r}+\frac{1}{2}\sigma^{2}r^{2}\frac{\partial^{2}B}{\partial r^{2}}-rB=0\\&B(T,T,r)=1 \qquad .\\&B(t,T,0)=1\end{aligned} \qquad (4.14)$$

By solving the PDE above, we can get the bond price. Now next we will use numerical methods to solve our equation.

We will use the same method to solve (4.14) as we did for (4.3). Since we already discussed in the first section of this chapter how to solve the PDE for CIR model, here we will simplify the procedure. For Dothan model, we have:

$$
\begin{aligned}
& ar\frac{\partial B}{\partial r}+\frac{1}{2}\sigma^2 r^2\frac{\partial^2 B}{\partial r^2}-rB \\
&= aih\frac{F_{j+1}^n-F_{j-1}^n}{2h}+\frac{1}{2}\sigma^2 i^2 h^2\frac{F_{j+1}^n-2F_j^n+F_{j-1}^n}{h^2}-ihF_j^n \\
&= \frac{(\sigma^2 i^2+ai)}{2}F_{j+1}^n+(-(ih+\sigma^2 i^2))F_j^n+\frac{(\sigma^2 i^2-ai)}{2}F_{j-1}^n \\
&= AF^n+b
\end{aligned}
$$

$(j=1.......N)$

where

$$
A=\begin{pmatrix}
-(ih+\sigma^2 i^2) & \frac{(\sigma^2 i^2+ai)}{2} & 0 \\
\frac{(\sigma^2 i^2-ai)}{2} & \ddots & \frac{(\sigma^2 i^2+ai)}{2} \\
0 & b_{N-1}-c_{N+1} & 2c_{N+1}+a_N
\end{pmatrix}_{N\times N}
$$

$$a_N=-(ih+\sigma^2 i^2)\,|_{i=N}$$

$$b_{N-1}=\frac{(\sigma^2 i^2-ai)}{2}\,|_{i=N-1}$$

$$c_{N+1}=\frac{(\sigma^2 i^2+ai)}{2}\,|_{i=N+1}$$

$$
b=\begin{pmatrix}
\frac{\sigma^2-a}{2} \\
0 \\
\vdots \\
0
\end{pmatrix}_{N\times 1}.
$$

We use Crank-Nicholson scheme to get the approximate solution, and now the equation will be changed as:

$$\frac{F^n-F^{n-1}}{dt}+\frac{1}{2}A(F^n+F^{n-1})+\frac{1}{2}(b^n+b^{n-1})=0$$

$$\Rightarrow(\frac{1}{2}A-\frac{I}{dt})F^{n-1}=-(\frac{1}{2}A+\frac{I}{dt})F^n-\frac{1}{2}(b^n+b^{n-1})$$

$$\Rightarrow A_1F^{n-1}=B_1F^n+b_1.$$

$$A_1 = (\frac{1}{2}A - \frac{I}{dt})$$
$$B_1 = -(\frac{1}{2}A + \frac{I}{dt})$$
$$b_1 = -\frac{1}{2}(b^n + b^{n-1})$$

Where A_1, B_1 are both $N \times N$ matrix, and I is $N \times N$ identity matrix, while N is the number of how many steps of the interest rate is divided. We use Matlab to implement our numerical solution, and we get the following results.

Parameters:

$\sigma = 0.3$
$a = 0.1$
$r_{\max} = 0.7$
$T = 3$

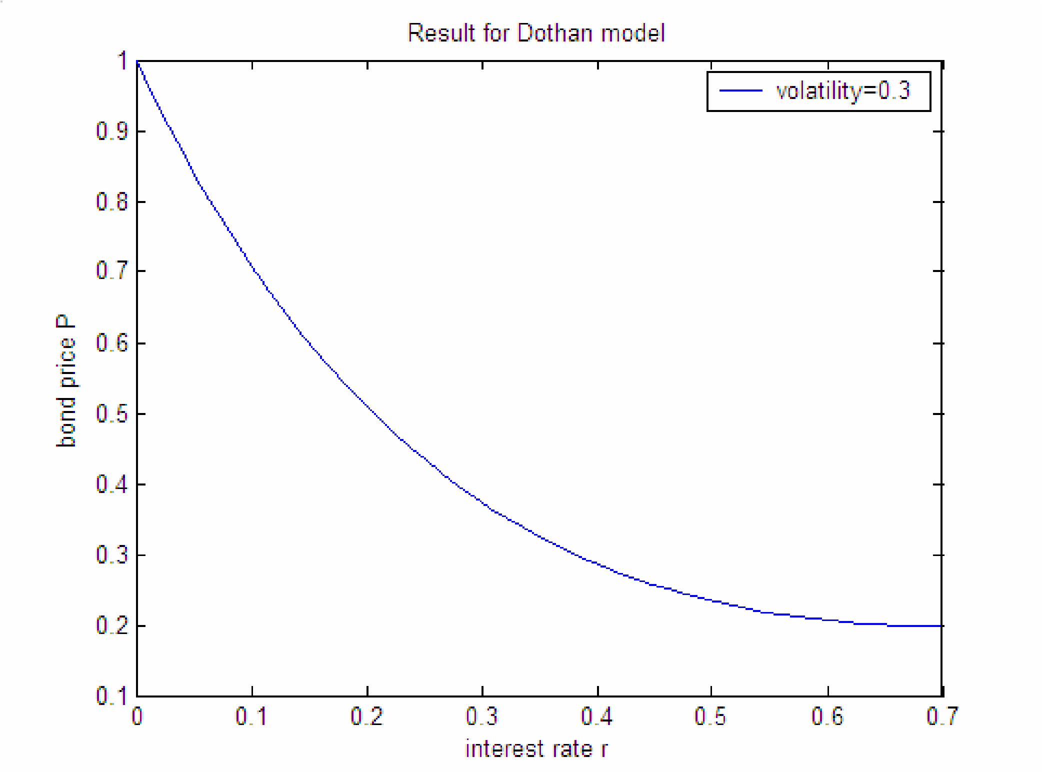

Figure 11 Result from finite difference methods for Dothan model

From Figure11, we can see the result is pretty nice, and we can easily see from the graph that bond price is convex as a function of interest rate r.

If we choose the same parameters, we will get the corresponding graph for logarithm of bond price.

Parameters:

$\sigma = 0.3$

$a = 0.1$

$r_{\max} = 0.7$

$T = 3$

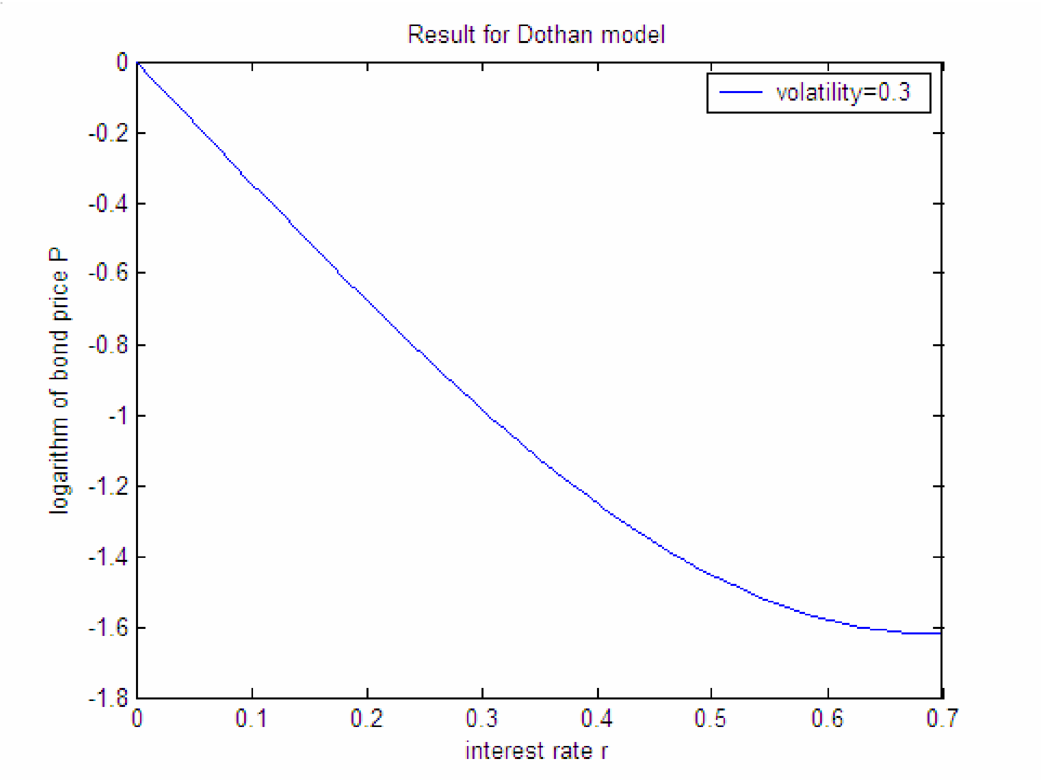

Figure 12 Logarithm of bond price for Dothan model

From Figure12, we can see that the logarithm of bond price is also convex as a function of interest rate r.

From the numerical solutions, we can also check if the bond price increases with the volatility in the Dothan short rate model or not. Here we choose different volatility for three groups of parameters, where we keep all the other parameters the same value except volatility.

Parameters:

$a = 0.1 \qquad \sigma_1 = 0.3$

$r_{\max} = 0.7 \qquad \sigma_2 = 0.5,$

$T = 3 \qquad \sigma_3 = 0.8$

and we get:

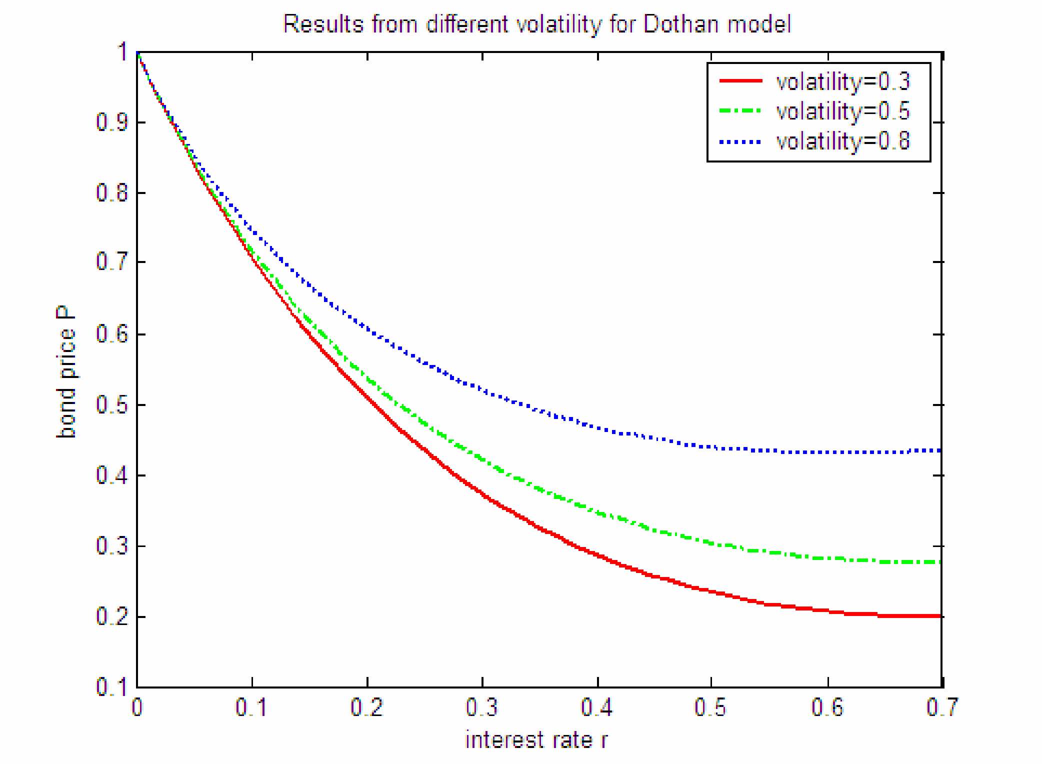

Figure 13 Results from different volatility for Dothan model

From Figure13, it is very easy to find out that the bond prices increase in the volatility of Dothan model.

If we keep all the parameters the same, we will get the corresponding graph for logarithm of bond prices for Dothan model.

Parameters:

$a = 0.1 \qquad \sigma_1 = 0.3$

$r_{\max} = 0.7 \qquad \sigma_2 = 0.5$

$T = 3 \qquad \sigma_3 = 0.8$

Graph:

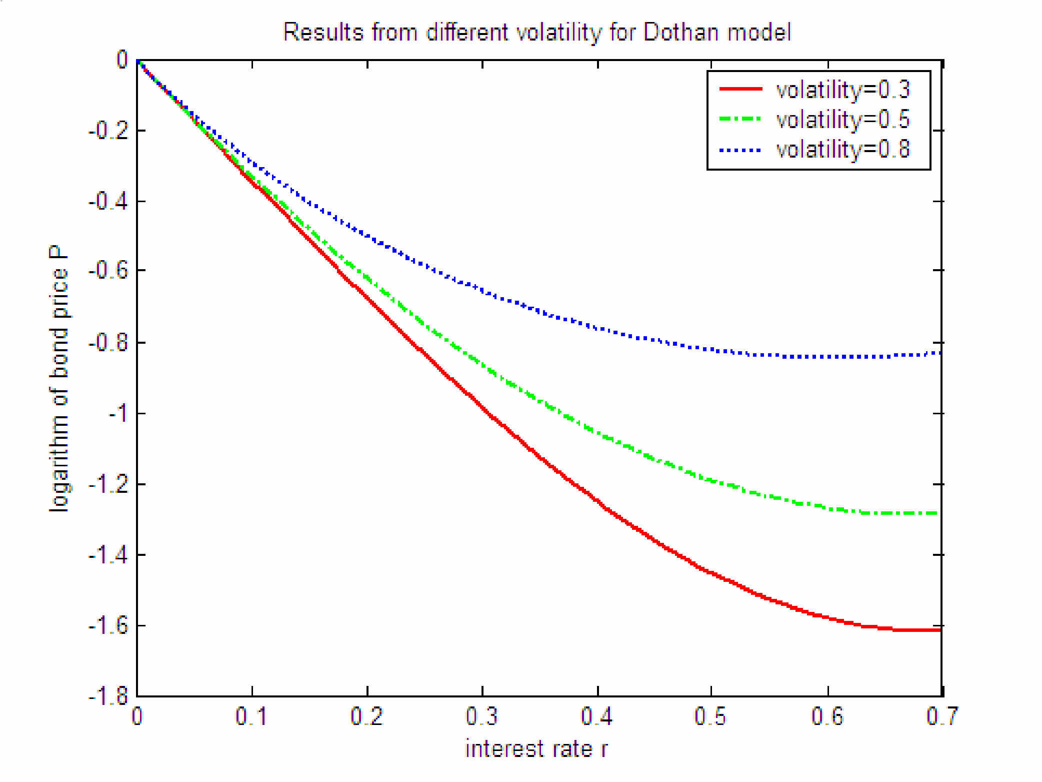

Figure 14 Logarithm of bond price with different volatility for Dothan model

From Fihure14, we can see that logarithm of the bond prices also increase in the volatility of Dothan model.

Chapter 5 Conclusion

In this paper, we discuss the convexity and monotonicity properties of bond prices and logarithms of bond prices for Cox, Ingersoll, and Ross model, Vasicek model and Dothan model. We use numerical method to get the solutions for these models, for Vasicek model , we also proved its convexity and monotonicity. Finally, we get the results summarized in Table 4 below. In the four last columns it is indicated which models preserve convexity for bond prices, log-convexity of bond prices, monotonicity of bond prices and log-monotonicity of bond prices.

Model	Dynamics	C	LCV	M	LM
Vasicek	$dr_t = k[\theta - r_t]dt + \sigma\, dW_t$	Yes	Yes	Yes	Yes
CIR	$dr_t = k[\theta - r_t]dt + \sigma \sqrt{r_t}\, dW_t$	Yes	Yes	Yes	Yes
Dothan	$dr_t = ar_t dt + \sigma\, r_t dW_t$	Yes	Yes	Yes	Yes

Table 4

Acknowledgements

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- Professor Per Lötstedt

- People who care about me

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