# Hand-out #3

1 Oct 08

Let $E$ be a subset of $\mathbb{C}$. Given $\sum_{n=1}^{\infty} f_n(z)$ for $z$ restricted to $E$. We say the infinite series has <u>majorized</u> (or <u>dominated</u>) convergence on $E$ when

$$|f_n(z)| \leq M_n \quad \text{for } n \geq 1 \text{ and } z \in E$$

**AND**

$$\sum_{n=1}^{\infty} M_n < +\infty .$$

$\boxed{M_n = \text{pos. constants}}$

Notice that majorized convergence <u>implies</u> absolute convergence!

<u>USEFUL THEOREM</u> (sometimes called Weierstrass' thm)

Let $B$ be any domain in the complex plane. Let $f_n(z)$ be <u>analytic</u> on $B$. Let $\sum_{n=1}^{\infty} f_n(z)$ have <u>majorized</u> convergence on $B$. Let $\gamma$ be any piecewise smooth path in $B$. Let $Q(z)$ be <u>continuous</u> on $\gamma$. Let $F(z) = \sum_{n=1}^{\infty} f_n(z)$. Then:

(a) $F(z)$ is continuous on $B$;

(b) $F(z)$ is analytic on $B$;

(c) $F^{(k)}(z) = \sum_{n=1}^{\infty} f_n^{(k)}(z)$ on $B$ (as a conv. series);

(d) $\int_{\gamma} F(z)\,Q(z)\,dz = \sum_{n=1}^{\infty} \int_{\gamma} f_n(z)\,Q(z)\,dz$ (as a conv. series).

Proof

(a). Choose any $z_0 \in B$ and any $\varepsilon > 0$. Select an $N$ so that $\sum_{n=N+1}^{\infty} M_n < \frac{\varepsilon}{10}$.

Notice that

$$|R_N(z)| \leqq \sum_{N+1}^{\infty} |f_n(z)| \leqq \sum_{N+1}^{\infty} M_n < \frac{\varepsilon}{10}$$

for $R_N(z) = \sum_{n=N+1}^{\infty} f_n(z)$. Notice also that

$$|F(z) - F(z_0)| = \left| \sum_{n=1}^{N} f_n(z) + R_N(z) - \sum_{n=1}^{N} f_n(z_0) - R_N(z_0) \right|$$

$$\leqq \left| \sum_{n=1}^{N} f_n(z) - \sum_{n=1}^{N} f_n(z_0) \right| + \frac{\varepsilon}{10} + \frac{\varepsilon}{10} .$$

Remember that $N$ is frozen. Thus, $\sum_{n=1}^{N} f_n(z)$ is certainly <u>continuous</u> on $B$. Choose $\delta$ so small that

$$\left| \sum_{n=1}^{N} f_n(z) - \sum_{n=1}^{N} f_n(z_0) \right| < \frac{\varepsilon}{10} \quad \text{for} \quad |z - z_0| < \delta .$$

Accordingly, for $|z - z_0| < \delta$ and $z \in B$,

$$|F(z) - F(z_0)| < \frac{\varepsilon}{10} + \frac{\varepsilon}{10} + \frac{\varepsilon}{10} < \varepsilon .$$

This proves $F$ is continuous at $z_0$. $\boxed{\text{OK}}$

<u>(d)</u> is next. Let $|Q(z)| \leqq \mathfrak{M}$ along curve $\gamma$. Let $L$ = length of $\gamma$. Must show that

$$\int_\gamma F(z) Q(z)\, dz = \lim_{N \to \infty} \sum_{n=1}^{N} \int_\gamma f_n(z) Q(z)\, dz .$$

So, choose any $\varepsilon > 0$. Choose $N_0$ so that

$$\sum_{n=N_0+1}^{\infty} M_n < \frac{\varepsilon}{2(L+1)(\mathfrak{M}+1)} .$$

Keep $N \geqq N_0$ and write $R_N(z) = \sum_{n=N+1}^{\infty} f_n(z)$ as before.

Notice that

$$\int_\gamma F(z)Q(z)\,dz - \sum_{n=1}^{N} \int_\gamma f_n(z)\,Q(z)\,dz$$

$$= \int_\gamma \Big[F(z) - \sum_{n=1}^{N} f_n(z)\Big] Q(z)\,dz$$

$$= \int_\gamma R_N(z)\,Q(z)\,dz \quad ,$$

so we get:

$$\left| \int_\gamma F(z)Q(z)\,dz - \sum_{n=1}^{N} \int_\gamma f_n(z)\,Q(z)\,dz \right|$$

$$\leqq \int_\gamma |R_N(z)|\,|Q(z)|\,ds \qquad \left\{ \begin{array}{l} \text{by } \Delta \text{ inequality} \\ \text{for integrals} \end{array} \right\}$$

$$\leqq \int_\gamma \Big( \sum_{N+1}^{\infty} |f_n(z)| \Big) \mathcal{M}\,ds$$

$$\leqq \int_\gamma \Big( \sum_{N+1}^{\infty} M_n \Big) \mathcal{M}\,ds$$

$$< \frac{\varepsilon}{2(L+1)(\mathcal{M}+1)} \mathcal{M} \cdot L < \frac{\varepsilon}{2} < \varepsilon$$

for all $N \geqq N_0$. This proves (d) for any path $\gamma$. $\boxed{\text{OK}}$

(b). Choose any $z_0$ in $B$. Select a small $h$ so that $\{|z-z_0|<h\} \subseteq B$. Call this disk $B_0$. Let $\gamma$ be any simple closed path in $B_0$. Take $Q(z) \equiv 1$. Apply (d).

Hence:

$$\oint_\gamma F(z)\,dz = \sum_{n=1}^{\infty} \oint_\gamma f_n(z)\,dz = \sum_{n=1}^{\infty} 0 = 0$$

by CIT. By Morera's theorem (S&S p.210), we conclude $F(z)$ must be analytic in $B_0$. Since $z_0$ is arbitrary, $F$ is analytic at each point of $B$; i.e., $F$ is analytic on $B$. This proves (b). $\boxed{OK}$

(c). We will prove this at each fixed point $z_0$ of $B$. Choose $h$ as in the proof of (b). Let $\gamma$ be the path $|z-z_0| = \frac{1}{2}h$. Let $Q(z)$ be $(z-z_0)^{-k-1}$ on $\gamma$. Apply (d) again. We get:

$$\oint_\gamma F(z)(z-z_0)^{-k-1}\,dz = \sum_{n=1}^{\infty} \oint_\gamma f_n(z)(z-z_0)^{-k-1}\,dz.$$

Apply CIF for derivatives. Get:

$$2\pi i \frac{F^{(k)}(z_0)}{k!} = \sum_{n=1}^{\infty} 2\pi i \frac{f_n^{(k)}(z_0)}{k!}.$$

Multiply thru by $\frac{k!}{2\pi i}$. We get

$$F^{(k)}(z_0) = \sum_{n=1}^{\infty} f_n^{(k)}(z_0) \quad \left(\text{as a } \begin{array}{c}\text{conv.}\\ \text{series}\end{array}\right)$$

for each $z_0$ in $B$. This proves (c). $\boxed{OK}$

QED.

We have been doing various differentiation and integration operations on power series and Laurent series in class. We can use Weierstrass' theorem to <u>justify</u> these operations once we verify that power series and Laurent series actually have <u>majorized</u> convergence.

To get that, one just has to "move in slightly" from the edge.

## FACT

Let $\sum_0^\infty A_n z^n$ converge on $|z| < R$. Choose any $r$ such that $0 < r < R$. The power series $\sum_0^\infty A_n z^n$ then has <u>majorized</u> convergence on $|z| \leqq r$.

Similarly, let $\sum_{-\infty}^\infty c_n z^n$ converge on $R_1 < |z| < R_2$. Choose $r_j$ so that $R_1 < r_1 < r_2 < R_2$. Then, $\sum_{-\infty}^{-1} c_n z^n$ and $\sum_0^\infty c_n z^n$ have <u>majorized</u> convergence on $r_1 \leqq |z| \leqq r_2$.

<u>Proof</u>

For $\sum_0^\infty A_n z^n$, choose any $\rho$ so that $r < \rho < R$. By hypothesis, $\sum_0^\infty A_n \rho^n$ converges. Therefore $\lim_{n\to\infty} A_n \rho^n = 0$. Therefore $\{A_n \rho^n\}_{n=0}^\infty$ is a bounded sequence. Let $|A_n\rho^n| \leqq \mathfrak{M}$, all $n$. So,

$$|A_n| \leqq \frac{\mathfrak{M}}{\rho^n}.$$

For $|z| \leqq r$, notice that $\boxed{\text{for } f_n(z) = A_n z^n}$

$$|A_n z^n| \leqq \mathfrak{M}\frac{|z|^n}{\rho^n} \leqq \mathfrak{M}\left(\frac{r}{\rho}\right)^n.$$

Take $M_n = \mathfrak{M}\left(\frac{r}{\rho}\right)^n$ and note that $\frac{r}{\rho} < 1$. $\boxed{\text{OK}}$

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For $\sum_{-\infty}^{\infty} c_n z^n$, take $R_1 < \rho_1 < r_1 < r_2 < \rho_2 < R_2$.

We know that $\sum_{-\infty}^{\infty} c_n \rho_1^n$ and $\sum_{-\infty}^{\infty} c_n \rho_2^n$ converge, by assumption. Therefore, certainly,

$$\sum_{-\infty}^{-1} c_n \rho_1^n \text{ converges}; \quad \sum_{0}^{\infty} c_n \rho_2^n \text{ converges}.$$

Therefore: $\lim_{n\to-\infty} c_n\rho_1^n = 0$, $\lim_{n\to\infty} c_n\rho_2^n = 0$.

Hence $|c_n\rho_1^n| \leqq \mathfrak{M}_1$, $|c_n\rho_2^n| \leqq \mathfrak{M}_2$ for all $n \leqq -1$, $n \geqq 0$, respectively. Get:

$$|c_n| \leqq \frac{\mathfrak{M}_1}{\rho_1^n} \quad \text{for } n \leqq -1$$

$$|c_n| \leqq \frac{\mathfrak{M}_2}{\rho_2^n} \quad \text{for } n \geqq 0.$$

For $r_1 \leqq |z| \leqq r_2$, we get: since $n<0$

$$|c_n z^n| \leqq \frac{\mathfrak{M}_1}{\rho_1^n}|z|^n = \mathfrak{M}_1\left(\frac{\rho_1}{|z|}\right)^{|n|} \leqq \mathfrak{M}_1\left(\frac{\rho_1}{r_1}\right)^{|n|}$$

for $n \leqq -1$;

$$|c_n z^n| \leq \frac{M_2}{\rho_2^n}|z|^n \leq M_2\left(\frac{r_2}{\rho_2}\right)^n \quad \text{for } n \geq 0.$$

Notice that $\frac{\rho_1}{r_1} < 1$, $\frac{r_2}{\rho_2} < 1$ ! We choose

$$M_n = \left\{ \begin{array}{ll} M_1\left(\frac{\rho_1}{r_1}\right)^{|n|} & \text{if } n \leq -1 \\ M_2\left(\frac{r_2}{\rho_2}\right)^n & \text{if } n \geq 0 \end{array} \right\}.$$

This choice of $M_n$ works. $\boxed{OK}$

QED

Example:

Suppose that $\sum_{-\infty}^{\infty} c_n z^n = \sum_{-\infty}^{\infty} d_n z^n$ (numerically) on $R_1 < |z| < R_2$. Choose $r_j$ so that $R_1 < r_1 < r_2 < R_2$. Let $\gamma$ be the circle $\{|z| = \frac{r_1+r_2}{2}\}$. Apply Weierstrass' theorem with $B = \{r_1 < |z| < r_2\}$, $Q(z) = z^{-m-1}$. $\boxed{m \text{ fixed}}$

Get:

$$\left[\begin{array}{l} \oint_\gamma \left(\sum_{-\infty}^{\infty} c_n z^n\right) z^{-m-1}\,dz = \sum_{-\infty}^{\infty} c_n \oint_\gamma z^{n-m-1}\,dz \\ \oint_\gamma \left(\sum_{-\infty}^{\infty} d_n z^n\right) z^{-m-1}\,dz = \sum_{-\infty}^{\infty} d_n \oint_\gamma z^{n-m-1}\,dz. \end{array}\right.$$

Therefore, $0 + 2\pi i c_m + 0 = 0 + 2\pi i d_m + 0$.

Hence: $c_m = d_m$ for every $m$. (EXACTLY AS EXPECTED.)