A TWISTED APPROACH TO KOSTANT'S PROBLEM

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1. Kostant's problem

$\mathfrak{g}$ — semi-simple finite-dimensional complex Lie algebra.

$U(\mathfrak{g})$ — the universal enveloping algebra of $\mathfrak{g}$.

M — $\mathfrak{g}$-module.

$\mathrm{Ann}(M)$ — the annihilator of M in $U(\mathfrak{g})$.

Then $U(\mathfrak{g})/\mathrm{Ann}(M) \hookrightarrow \mathrm{End}_{\mathbb{C}}(M)$.

$U(\mathfrak{g})$ is a $U(\mathfrak{g}) - U(\mathfrak{g})$-bimodule.

Every $U(\mathfrak{g})-U(\mathfrak{g})$-bimodule M is a $\mathfrak{g}$-module via the *adjoint action* $x\cdot m = xm - mx$.

An $U(\mathfrak{g})-U(\mathfrak{g})$-bimodule is called a *Harish-Cahndra* bimodule if the adjoint action of $\mathfrak{g}$ on it is locally finite.

$U(\mathfrak{g})$ is a Harish-Cahndra $U(\mathfrak{g})-U(\mathfrak{g})$-bimodule.

Let M, N be two $\mathfrak{g}$-modules.

Then $\mathrm{Hom}_{\mathbb{C}}(M,N)$ is a $U(\mathfrak{g})-U(\mathfrak{g})$-bimodule.

$U(\mathfrak{g})/\mathrm{Ann}(M)\hookrightarrow \mathrm{End}_{\mathbb{C}}(M)$ is a bimodule homomorphism.

$\mathscr{L}(M,N)$ — the maximal Harish-Cnandra subbimodule of the bimodule $\mathrm{Hom}_{\mathbb{C}}(M,N)$.

$U(\mathfrak{g})/\mathrm{Ann}(M)\hookrightarrow \mathscr{L}(M,M)$.

Kostant's Problem: For which simple $U(\mathfrak{g})$-modules M do we have $U(\mathfrak{g})/\mathrm{Ann}(M)\cong \mathscr{L}(M,M)$?

There is no complete answer even for simple highest weight modules.

2. Category $\mathcal{O}$ and twisting functors

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$.

For $\lambda \in \mathfrak{h}^*$ define:

- $M(\lambda)$ — Verma module with highest weight λ;
- $L(\lambda)$ — the unique simple quotient of $M(\lambda)$.

$\mathcal{O}$ — BGG category $\mathcal{O}$ for $\mathfrak{g}$.

$\mathcal{O}_0$ — the principal block of $\mathcal{O}$.

W — the Weyl group of $\mathfrak{g}$.

dot-action: $w \cdot \lambda = w(\lambda + \rho) - \rho$.

For $w \in W$ set:

- $L(w) = L(w \cdot 0)$;
- $\Delta(w) = M(w \cdot 0)$;
- $P(w)$ — the indecomposable projective cover of $L(w)$;
- $\nabla(w)$ is the $\mathcal{O}$-dual of $\Delta(w)$;
- θ_w — the indecomposable translation functor on $\mathcal{O}_0$.
- $\mathtt{T}_w$ — the twisitng functor on $\mathcal{O}_0$.

$\{L(w) : w \in W\}$ — a complete set of simples in $\mathcal{O}_0$.

Properties of $\mathtt{T}_w$:

[AL] *H. H. Andersen, N. Lauritzen*, Twisted Verma modules. Studies in memory of Issai Schur (Chevaleret/Rehovot, 2000), 1–26, Progr. Math., 210, Birkhuser Boston, Boston, MA, 2003.

[AS] *H. H. Andersen, C. Stroppel*, Twisting functors on $\mathcal{O}$. Represent. Theory 7 (2003), 681–699.

[Kh] *O. Khomenko*, Categories with projective functors, Ph.D. Thesis, Freiburg University, Freiburg, Germany, 2003.

[KM] *O. Khomenko, V. Mazorchuk*, On Arkhipov's and Enright's functors, Math. Z. 249 (2005), 357–386.

[MS] *V. Mazorchuk, C. Stroppel*, On functors associated with simple roots, preprint math.RT/0410339.

(I) $\mathtt{T}_w\theta_x \cong \theta_x\mathtt{T}_w$, $\forall w, x \in W$, [AS, Theorem 3.2].

(II) $\mathcal{L}_i\mathtt{T}_w\Delta(x) = 0$, $\forall w, x \in W$, $\forall i > 0$, [AS, Theorem 2.2].

(III) $\mathcal{L}\mathtt{T}_w : \mathcal{D}^b(\mathcal{O}_\lambda) \xrightarrow{\sim} \mathcal{D}^b(\mathcal{O}_\lambda)$, [AS, Corollary 4.2].

(IV) $\mathtt{T}_w \cong \mathtt{T}_{s_1} \cdot \dots \cdot \mathtt{T}_{s_k}$, if $w = s_1 \dots s_k$ is reduced, [AS,Lemma 2.1], [KM, Corollary 11].

(V) $\mathtt{T}_s \Delta(x) \cong \Delta(sx)$, $sx > x$, [AL, Lemma 6.2];

(VI) [AS, Theorem 2.3]

$$\mathtt{T}_s \nabla(x) \cong \begin{cases} \nabla(x), & x < sx, \\ \nabla(sx), & x > sx, \end{cases}$$

(VII) $\mathtt{T}_s L(x) \neq 0$ iff $sx < x$, [AS, Section 6].

(VIII) $\mathcal{L}_1\mathtt{T}_s$ is s-Zuckermann functor (i.e. taking of the largest s-finite submodule of M), [MS, Theorem 1] or [Kh, Proposition 6].

(IX) $\mathtt{T}_s$ is left adjoint to $\mathtt{G}_s = \star\mathtt{T}_s\star$, where $\star$ is the duality on $\mathcal{O}$, [AS, Theorem 4.1], [KM, Corollary 6].

(X) Inverse in (III) is $\mathcal{R}\mathtt{G}_{w^{-1}}$, [AS, Corollary 4.2].

3. Classical results

Statement 1. $M, N, V \in \mathfrak{g}$-mod, $\dim V < \infty$. Then there are canonical isomorphisms

$$\mathrm{Hom}_{\mathfrak{g}}(V, \mathscr{L}(M, N)) \cong \mathrm{Hom}_{\mathfrak{g}}(M \otimes V, N) \cong \mathrm{Hom}_{\mathfrak{g}}(M, N \otimes V^*).$$

Statement 2.

$$U(\mathfrak{g})/\mathrm{Ann}(\Delta(e)/M) \cong \mathscr{L}(\Delta(e)/M, \Delta(e)/M) \qquad \forall M \subset \Delta(e).$$

Statement 3.

$$\mathrm{Ann}(\Delta(w)) = \mathrm{Ann}(\Delta(e)) \qquad \forall w \in W.$$

Theorem. (Joseph)

$$U(\mathfrak{g})/\mathrm{Ann}(\Delta(w)) \cong \mathscr{L}(\Delta(w), \Delta(w)).$$

Proof.

Since $\Delta(w) \subset \Delta(e)$, using Statement 2 and 3 we only need

$$\dim \mathrm{Hom}_{\mathfrak{g}}(V, \mathscr{L}(\Delta(w), \Delta(w))) = \dim \mathrm{Hom}_{\mathfrak{g}}(V, \mathscr{L}(\Delta(e), \Delta(e)))$$

for every finite-dimensional $V \in \mathfrak{g}$-mod.

$$\begin{aligned}
&\mathrm{Hom}_{\mathfrak{g}}(V, \mathscr{L}(\Delta(w), \Delta(w))) &&= \text{Statement 1}\\
&\mathrm{Hom}_{\mathfrak{g}}(\Delta(w), \Delta(w) \otimes V^*) &&= \text{(IV) and (V)}\\
&\mathrm{Hom}_{\mathfrak{g}}(\mathrm{T}_w\Delta(e), \mathrm{T}_w(\Delta(e)) \otimes V^*) &&= \text{(I)}\\
&\mathrm{Hom}_{\mathfrak{g}}(\mathrm{T}_w\Delta(e), \mathrm{T}_w(\Delta(e) \otimes V^*)) &&=\\
&\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O}_\lambda)}(\mathrm{T}_w\Delta(e), \mathrm{T}_w(\Delta(e) \otimes V^*)) &&= \text{(II)}\\
&\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O}_\lambda)}(\boldsymbol{\mathcal{L}}\mathrm{T}_w\Delta(e), \boldsymbol{\mathcal{L}}\mathrm{T}_w(\Delta(e) \otimes V^*)) &&= \text{(III)}\\
&\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O}_\lambda)}(\Delta(e), \Delta(e) \otimes V^*) &&=\\
&\mathrm{Hom}_{\mathfrak{g}}(\Delta(e), \Delta(e) \otimes V^*) &&= \text{Statement 1}\\
&\mathrm{Hom}_{\mathfrak{g}}(V, \mathscr{L}(\Delta(e), \Delta(e))).
\end{aligned}$$

Q.E.D.

3. Main result

$\varnothing \neq S$ — a set of simple roots.

W^S — the corresponding Weyl group.

w_0 — the longest element in W.

w_0^S — the longest element in W^S.

Theorem. (Gabber-Joseph)

$$U(\mathfrak{g})/\mathrm{Ann}(L(w_0^S w_0)) \cong \mathscr{L}(L(w_0^S w_0), L(w_0^S w_0)).$$

s — a simple reflection from W^S.

Main Result.

$$U(\mathfrak{g})/\mathrm{Ann}(L(sw_0^S w_0)) \cong \mathscr{L}(L(sw_0^S w_0), L(sw_0^S w_0)).$$

Proof in the case $W_S = W$.

Statement. $s \in W$ — simple reflection. Then

$$U(\mathfrak{g})/\mathrm{Ann}(L(s)) \cong \mathscr{L}(L(s), L(s)).$$

Lemma 1. Let $0 \to X \to \Delta(w) \to Y \to 0$ be a s.e.s. such that $\mathrm{Ext}^1_{\mathcal{O}}(\Delta(w), X \otimes V) = 0$ for every finite-dimensional $\mathfrak{g}$-module V. Then

$$U(\mathfrak{g})/\mathrm{Ann}(Y) \cong \mathscr{L}(Y, Y).$$

Proof. Apply $\mathrm{Hom}_{\mathfrak{g}}(\Delta(w), \_ \otimes V)$ to s.e.s.

From ext-vanishing we have $\mathscr{L}(\Delta(w), \Delta(w)) \twoheadrightarrow \mathscr{L}(\Delta(w), Y)$.

$\mathscr{L}(Y, Y)$ is a subspace of $\mathscr{L}(\Delta(w), Y)$. **Q.E.D.**

Lemma 2. $\mathrm{Hom}_{\mathfrak{g}}(L(s), \theta_w L(s)) \neq 0$ implies $w = s$ or $w = e$.

Proof.

$w \neq e, s$. Consider $0 \to L(s) \to X \to L(e) \to 0$, non-split.

$\theta_w L(e) = 0$ since $w \neq e$. Hence $\theta_w X = \theta_w L(s)$.

Since $\Delta(e) \twoheadrightarrow X$, we have $\theta_w \Delta(e) = P(w) \twoheadrightarrow \theta_w X$.

Thus either $\theta_w X = 0$ or $\mathrm{top}(\theta_w X) = L(w)$.

$\theta_w X$ is self-dual.

Therfore either $\theta_w X = 0$ or $\mathrm{Soc}(\theta_w X) = L(w)$.

In either case $\mathrm{Hom}_{\mathfrak{g}}(L(s), \theta_w L(s)) = 0$. **Q.E.D.**

Consider

$$0 \to F(s) \to \Delta(s) \to N(s) \to 0,$$

where $F(s)$ is the minimal submodule of $\mathrm{Rad}(\Delta(s))$ such that the quotient $\mathrm{Rad}(\Delta(s))/F(s)$ is s-finite.

Lemma 3.

$$U(\mathfrak{g})/\mathrm{Ann}(N(s)) \cong \mathscr{L}(N(s), N(s)).$$

Proof.

$$
\begin{array}{lll}
\mathrm{Ext}^1_{\mathcal{O}}(\Delta(s), \theta_w F(s)) & = & \\
\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\Delta(s), \theta_w F(s)[1]) & = & \text{(V)} \\
\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\mathtt{T}_s\Delta(e), \theta_w F(s)[1]) & = & \text{(properties of } \theta_w\text{)} \\
\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\theta_{w^{-1}}\mathtt{T}_s\Delta(e), F(s)[1]) & = & \text{(I)} \\
\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\mathtt{T}_s\theta_{w^{-1}}\Delta(e), F(s)[1]) & = & \\
\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\mathtt{T}_s P(w^{-1}), F(s)[1]) & = & \text{(II)} \\
\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\mathcal{L}\mathtt{T}_s P(w^{-1}), F(s)[1]) & = & \text{(III)} \\
\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(P(w^{-1}), \mathcal{R}\mathtt{G}_s F(s)[1]). & = & (P(w^{-1}) \text{ is projective}) \\
\mathrm{Hom}_{\mathfrak{g}}(P(w^{-1}), \mathcal{R}^1\mathtt{G}_s F(s)) & = & \text{(dual of (VIII))} \\
0. & &
\end{array}
$$

The statement now follows from Lemma 1 and Joseph's Theorem.

Lemma 4. For every f.dim V

$$0 \to X(s) \to N(s) \to L(s) \to 0$$

induces an siomorphism

$$\mathrm{Hom}_{\mathfrak{g}}(N(s), N(s) \otimes V) \cong \mathrm{Hom}_{\mathfrak{g}}(L(s), L(s) \otimes V).$$

Proof.

$N(s)$ has simple top $L(s)$ and some s-finite junk, which remains s-finite after translations.

If $0 \neq f \in \mathrm{Hom}_{\mathfrak{g}}(N(s), N(s) \otimes V)$, f does not annihilate the top.

Hence $\mathrm{Hom}_{\mathfrak{g}}(N(s), N(s) \otimes V) \hookrightarrow \mathrm{Hom}_{\mathfrak{g}}(L(s), L(s) \otimes V)$.

Have to compare dimensions.

Need: $\dim \mathrm{Hom}_{\mathfrak{g}}(N(s), \theta_w N(s)) = \dim \mathrm{Hom}_{\mathfrak{g}}(L(s), \theta_w L(s))$.

$w = e, s$ — clear. $w \neq e, s$ — both zero by Lemma 2. **Q.E.D.**

Proof of the main statement.

$X(s)$ is s-finite, $L(s)$ is simple and s-infinite.

Hence $\mathscr{L}(X(s), L(s)) = 0$, implying

$$\mathscr{L}(L(s), L(s)) \cong \mathscr{L}(N(s), L(s)).$$

$X(s)$ is s-finite and $\mathrm{top}(N(s))$ is simple and s-infinite.

Hence $\mathscr{L}(N(s), X(s)) = 0$, implying

$$\mathscr{L}(N(s), N(s)) \hookrightarrow \mathscr{L}(N(s), L(s)).$$

Thus $\mathscr{L}(N(s), N(s)) \cong \mathscr{L}(L(s), L(s))$ by Lemma 4 and Statement 1.

$U(\mathfrak{g}) \twoheadrightarrow \mathscr{L}(N(s), N(s))$ by Lemma 3, **Q.E.D.**