Validated Numerics Methods for Mixed Boundary Value Problems for the System of Elastostatics

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The Dirichlet Problem

Let Ω be a bounded domain in $\mathbb{R}^n$ and $\mathcal{L}$ a second order homogeneous elliptic differential operator. The Dirichlet problem reads:

$$(D) \quad \begin{cases} \mathcal{L}u = 0 \quad \text{in} \quad \Omega \\ u\Big|_{\partial\Omega}^{n.t.} = f \quad \text{on} \quad \partial\Omega \\ \mathcal{N}(u) \in L^p(\partial\Omega) \end{cases}$$

For instance, $\mathcal{L} = \triangle$ (the case of the scalar Laplacian) or with $\mu > 0$ and $\lambda \geq -\frac{2\mu}{n}$:

$$\mathcal{L}\vec{u} = \mu\triangle\vec{u} + (\lambda + \mu)\nabla \mathrm{div}\, \vec{u},$$

- (D) well posed $\forall p \in (2 - \epsilon, +\infty)$ – sharp in the class of Lipschitz domains
- (D) can be reduced to a BIE of the type $(I + K)g = f$, where K is a singular integral operator of Calderón-Zygmund type

Non-tangential condition

Definition (Non-tangential approach region)

Fix $a > 1$ and for $x \in \partial\Omega$, consider the non-tangential approach region.

$$\Gamma_a(x) := \{y \in \Omega : |x - y| < a \cdot dist(y, \partial\Omega)\}$$

Remarks:

- When $\Omega = \mathbb{R}^n_+$, $\Gamma_a(x)$ is an infinite upright cone with vertex at x.
- a determines the aperture of the cone/non-tangential region.

Fix $a > 1$ and $u : \Omega \longrightarrow \mathbb{R}$.

Definition (Non-tangential Maximal Function and Limits)

The non-tangential maximal function of u at $x \in \partial\Omega$ and the non-tangential limit of u at $x \in \partial\Omega$ are:

$$\mathcal{N}(u)(x) := \sup_{y \in \Gamma_a(x)} |u(y)| \quad \text{and} \quad u\Big|_{\partial\Omega}^{n.t.}(x) := \lim_{\substack{y \to x \\ y \in \Gamma_a(x)}} u(y)$$

On the condition $\mathcal{N}(u) \in L^p(\partial\Omega)$

The condition $\mathcal{N}(u) \in L^p(\partial\Omega)$ is necessary and natural even in the case of the Dirichlet problem in the upper half space $\mathbb{R}^n_+$. Let $n = 2$ and $p \in (1, \infty)$ and consider

$$\begin{cases} \triangle u = 0 \quad \text{in} \quad \mathbb{R}^2_+ \\ u\Big|^{n.t.}_{\partial\mathbb{R}^2_+} = 0 \in L^p(\partial\mathbb{R}^2_+) \quad \text{on} \quad \partial\mathbb{R}^2_+ \end{cases}$$

Then

1. $u_1(x, y) \equiv 0$ is obviously one solution,
2. $u_2(x, y) = \dfrac{y}{x^2 + y^2}$ is a solution as well, violating uniqueness.

This happens because $\mathcal{N}\left(\dfrac{y}{x^2 + y^2}\right) \notin L^p(\partial\mathbb{R}^2_+)$, for any $p \in (1, \infty)$

The Neumann Problem

Fix $p \in (1, \infty)$. The Neumann problem with data in $L^p(\partial\Omega)$:

$$(N) \quad \begin{cases} \mathcal{L}u = 0 \quad \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega}^{n.t.} = f \in L^p(\partial\Omega) \quad \text{on} \quad \partial\Omega \\ \mathcal{N}(\nabla u) \in L^p(\partial\Omega) \end{cases}$$

When $\mathcal{L} = \Delta$ then $\dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega}^{n.t.} = \langle \nabla u\big|_{\partial\Omega}^{n.t.}, \nu \rangle$. For the Lamé system $\dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega}^{n.t.}$ stands for a so-called conormal derivative of u - infinitely many choices! We shall be working with:

$$\boxed{\frac{\partial \vec{u}}{\partial \nu}\Big|_{\partial\Omega}^{n.t.} := \mu \nabla \vec{u} \cdot \nu + \frac{\mu(\mu+\lambda)}{3\mu+\lambda} (\nabla \vec{u})^t \cdot \nu + \frac{(2\mu+\lambda)(\mu+\lambda)}{3\mu+\lambda} (\mathrm{div} \vec{u}) \nu}$$

- (N) is well-posed $\forall p \in (1, 2+\epsilon)$ and this is sharp in the class of Lipschitz domains

The Mixed Boundary Value Problem

Consider next the mixed boundary value problem (MBVP), also known as the Zaremba problem when $\mathcal{L} = \Delta$. Let Ω be a bounded domain in $\mathbb{R}^n$ with the splitting :

$$\partial\Omega = \bar{D} \cup \bar{N} \qquad \bar{D} \cap \bar{N} = \partial D = \partial N$$

Then the MBVP with L^p data, $p \in (1, \infty)$, is:

$$(MBVP) \quad \begin{cases} \mathcal{L}u = 0 \quad \text{in} \quad \Omega \\ u\Big|_D^{n.t.} = f \in L_1^p(D) \quad \text{on} \quad D \\ \dfrac{\partial u}{\partial \nu}\Big|_N^{n.t.} = g \in L^p(N) \quad \text{on} \quad N \\ \mathcal{N}(\nabla u) \in L^p(\partial\Omega) \end{cases}$$

A few remarks on the (*MBVP*)

- If $D = \emptyset$ then, (*MBVP*) becomes (*N*).
- If $N = \emptyset$ then, (*MBVP*) becomes (*D*) (with data in $L^p_1(\partial\Omega)$ - also known as the Regularity problem).
- This is an open problem in C. Kenig's 1994 *CBMS Book*: *characterize the smoothness of the gradient of the solution of (MBVP)*.
- The study of (*MBVP*) has applications in engineering and mathematical physics. Indeed, this has connections with
  1. conductivity
  2. heat transfer
  3. wave phenomena
  4. electrostatics
  5. metallurgical melting
  6. stamp problems in elasticity and hydrodynamics

Smooth domains for $(MBVP)$

Consider: $\Omega = \{z \in \mathbb{C} : 0 < \arg z < \alpha\} \cap \{|z| < 1\}$
with the splitting : $N = \{z = r\, e^{i\alpha}; 0 < r < 1\}, \quad D = \partial\Omega \setminus \overline{N}$

Let $(x, y) \equiv z = re^{i\theta}$ and $u(x, y) = r^{\frac{\pi}{2\alpha}} \sin\left(\frac{\pi\theta}{2\alpha}\right)$

1. $\Delta u = 0$ in Ω $\left(\text{since } u = \mathrm{Im}\left(z^{\pi/2\alpha}\right)\right)$
2. $u\Big|_D^{n.t} \in L^2_1(D)$
3. $\frac{\partial u}{\partial \nu}\Big|_N^{n.t.} \in L^2(N)$
4. if $\alpha \geq \pi$ then $\nabla u \notin L^2(\partial\Omega)$. In particular if $\alpha = \pi$ (i.e. in the smooth case, one does not have an L^2 theory)

Indeed $\nabla u\Big|_{\partial\Omega}^{n.t.} \approx r^{\frac{\pi}{2\alpha}-1}$. To integrate this around the origin on $\partial\Omega$ requires $\frac{\pi}{\alpha} - 2 > -1$, i.e.

$$\boxed{\alpha < \pi}$$

Layer Potentials

Let Γ be such that $\mathcal{L}\Gamma = \delta$. Introduce the single layer potential:

$$\mathcal{S}\vec{g}(X) := \int_{\partial\Omega} \Gamma(X - Q)\vec{g}(Q)\, d\sigma(Q) \qquad X \in \mathbb{R}^n \setminus \partial\Omega$$

Boundary behavior of $\mathcal{S}$ and $\frac{\partial \mathcal{S}}{\partial \nu}$:

- $\left.\dfrac{\partial \mathcal{S}\, \vec{g}}{\partial \nu}\right|_{\partial\Omega}^{n.t.} = \left(-\frac{1}{2}I + K^*\right)\vec{g}$
- $\left.\mathcal{S}\, \vec{g}\right|_{\partial\Omega}^{n.t.} = S\, \vec{g}$

Here, for $P \in \partial\Omega$

$$S\vec{g}(P) := \int_{\partial\Omega} \Gamma(P - Q)\vec{g}(Q)\, d\sigma(Q),$$

and

$$K^*\vec{g}(P) := p.v. \int_{\partial\Omega} \frac{\partial \Gamma}{\partial \nu(P)}(Q - P)\vec{g}(Q)\, d\sigma(Q)$$

The Case of a Sector

Ω -sector of aperture $\theta \in (0, 2\pi)$ and D/N be the left/right rays of $\partial\Omega$

Seek a solution of the (*MBVP*) as $\vec{u} = \mathcal{S}\vec{h}$ for some $\vec{h} : \partial\Omega \to \mathbb{R}^2$. Then

- $\mathcal{L}u \equiv 0$ in Ω
- $\vec{u}\Big|_D^{n.t.} = S\vec{h}\Big|_D = \vec{f} \in (L_1^p(D))^2$. Consequently
  $\partial_\tau S\vec{h}\Big|_D = \partial_\tau \vec{f} \in (L^p(D))^2$
- $\dfrac{\partial \vec{u}}{\partial \nu}\Big|_N^{n.t.} = (-\frac{1}{2}I + K^*)\vec{h}\Big|_N$

This leads us to consider $T : (L^p(\partial\Omega))^2 \to (L^p(D))^2 \oplus (L^p(N))^2$ with

$$T(\vec{h}) = \left(\partial_\tau S\vec{h}|_D, (-\tfrac{1}{2}I + K^*)\vec{h}\Big|_N\right)$$

The BIE satisfied by $\vec{h}$ is:

$$\left(\partial_\tau S\vec{h}|_D, (-\tfrac{1}{2}I + K^*)\vec{h}\Big|_N\right) = (\partial_\tau \vec{f}, \vec{g}) \quad \text{i.e.} \quad T(\vec{h}) = (\partial_\tau \vec{f}, \vec{g})$$

Hardy Kernels

Will need the notion of Hardy Kernel:

Definition (Hardy Kernel)

Let $k(\cdot,\cdot)$ be a measurable function on $\mathbb{R}_+ \times \mathbb{R}_+$. Then k is a Hardy kernel for $L^p(\mathbb{R}_+)$ provided that

1. $k(\cdot,\cdot)$ is homogenous of degree -1, and
2. $\int_0^\infty |k(1,y)| y^{-1/p} dy < \infty$

If $k(x,y)$ is a Hardy Kernel, define the Hardy Kernel operator $\mathscr{K}$ as

$$\mathscr{K} f(x) = \int_0^\infty k(x,y) f(y) dy$$

Theorem [Fabes, Jodeit, Lewis, Boyd]

Let h be a Hardy kernel on $(L^p(\mathbb{R}_+))^m$, for some $p \in (1, \infty)$, and let A, B be $m \times m$ matrices with real entries, and $c_1, c_2, c_3 \in \mathbb{R}$ given constants. Consider the operator $T : (L^p(\mathbb{R}_+))^m \longrightarrow (L^p(\mathbb{R}_+))^m$ given by

$$\boxed{Tf(s) := Af + \int_0^\infty \mathfrak{K}(s,t)f(t)\,dt} \quad \text{for a.e. } \; s \in \mathbb{R}_+ \;\&\; \forall\, f \in (L^p(\mathbb{R}_+))^m.$$

$$\text{with} \quad \boxed{\mathfrak{K}(s,t) := h(s,t) + \frac{1}{s-t} \cdot B}, \qquad \forall\, s,t \in \mathbb{R}_+ \;\; \text{with} \;\; s \neq t,$$

Then T is a linear bounded operator on $(L^p(\mathbb{R}_+))^m$and its spectrum is

$$\sigma(T; (L^p(\mathbb{R}_+))^m) = \overline{\{w \in \mathbb{C} : \det(wI - A - \mathcal{M}\mathfrak{K}(\cdot, 1)(1/p + i\xi)) = 0, \xi \in \mathbb{R}\}}$$

where I is the identity operator, $\mathcal{M}$ denotes the Mellin Transform, and $\overline{E}$ denotes the closure of the set $E \subseteq \mathbb{C}$ in $\mathbb{R}^2$.

Corollary

Corollary $(*)$

Consider the operator T as in the previous Theorem, such that

$$\det(A - \pi i \cdot B) \neq 0.$$

Then T is invertible on $(L^p(\mathbb{R}_+))^m$, $1 < p < \infty$, if and only if the following holds

$$\boxed{\det\left(A + \mathcal{M}(\mathfrak{K}(\cdot, 1))(1/p + i\xi)\right) \neq 0 \quad \forall \xi \in \mathbb{R}.}$$

When $\Omega \subseteq \mathbb{R}^2$ is a sector we let $\partial\Omega_1$ and $\partial\Omega_2$ denote the left and right rays of $\partial\Omega$. The idea is to identify $\partial\Omega_j, j = 1, 2$ with $\mathbb{R}_+$ via

$$\partial\Omega_j \ni P \mapsto |P| \in \mathbb{R}_+$$

Then, T has the structure from $(*)$ and this provides a mechanism to compute the critical values p as stated in the Theorem.

Laplacian

Theorem (Awala,Mitrea,Ott,2016)

Let Ω as before and $\mathcal{L} = \Delta$. Then the operator T is an isomorphism for all $p \in (1, \infty)$ s.t.

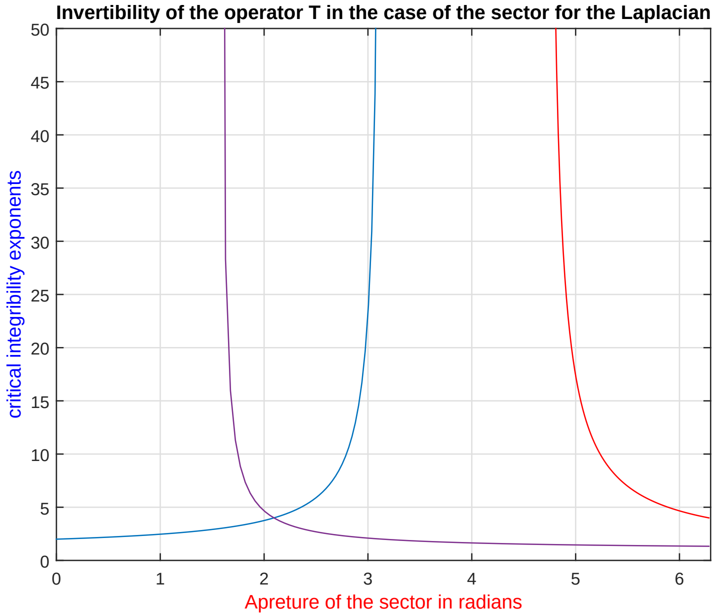
$$p \neq \begin{cases} \frac{2\pi-\theta}{\pi-\theta} & \text{if } \theta \in (0, \pi/2) \\ \frac{2\pi-\theta}{\pi-\theta}, \frac{2\theta}{2\theta-\pi} & \text{if } \theta \in (\pi/2, \pi) \\ \frac{2\theta}{2\theta-\pi} & \text{if } \theta \in (\pi, 3\pi/2) \\ \frac{2\theta}{2\theta-\pi}, \frac{2\theta}{2\theta-3\pi} & \text{if } \theta \in (3\pi/2, 2\pi) \\ 3 & \text{if } \theta = \pi/2 \\ 3/2, 3 & \text{if } \theta = 3\pi/2 \\ 2 & \text{if } \theta = \pi \end{cases}$$

Laplacian

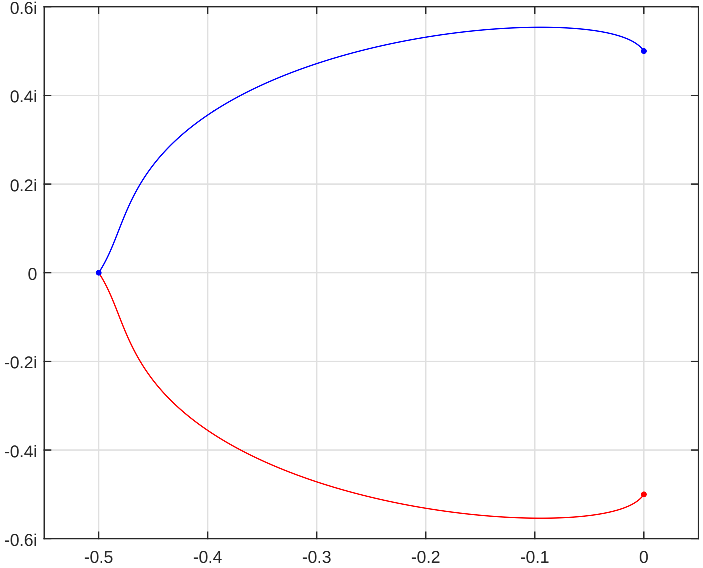
Theorem (Awala,Mitrea,Ott,2016)

Let Ω as before and $\mathcal{L} = \Delta$. Then, the mixed boundary value problem $(MBVP_p)$ is well-posed whenever

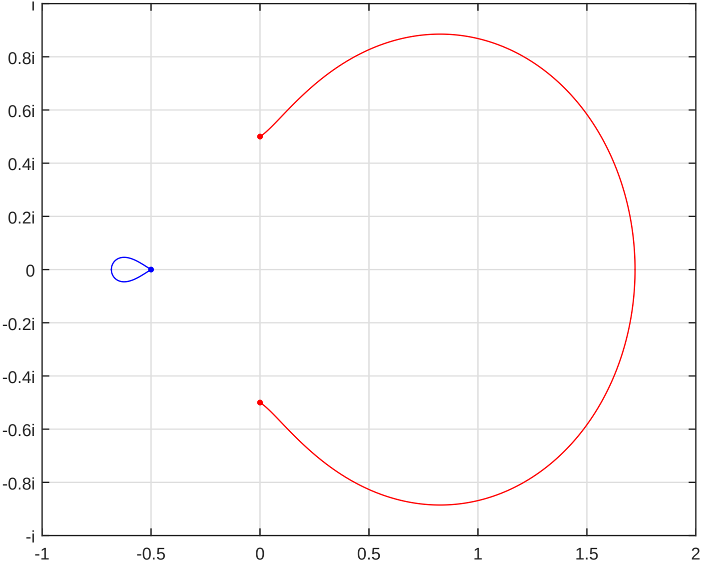
$$p \neq \begin{cases} \frac{2\pi-\theta}{\pi-\theta} & \text{if } \theta \in (0, \pi/2] \\ \frac{2\pi-\theta}{\pi-\theta}, \frac{2\theta}{2\theta-\pi} & \text{if } \theta \in (\pi/2, \pi) \\ 2 & \text{if } \theta = \pi \\ \frac{2\theta}{2\theta-\pi}, \frac{\theta}{\theta-\pi} & \text{if } \theta \in (\pi, 3\pi/2] \\ \frac{2\theta}{2\theta-\pi}, \frac{2\theta}{2\theta-3\pi}, \frac{\theta}{\theta-\pi} & \text{if } \theta \in (3\pi/2, 2\pi)\,. \end{cases}$$


Invertibility of the operator T in the case of the sector for the Laplacian
critical integribility exponents
Apreture of the sector in radians
0
5
10
15
20
25
30
35
40
45
50
0
1
2
3
4
5
6

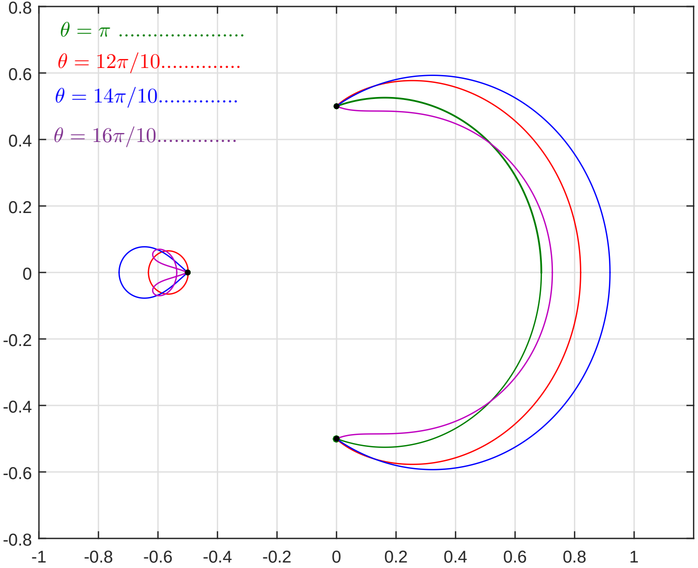
$\sigma\big(T; L^{1.5}(D) \oplus L^{1.5}(N)\big)$ when $\theta = \frac{\pi}{2}$



$\sigma\big(T; L^{10}(D) \oplus L^{10}(N)\big)$ when $\theta = \frac{3\pi}{2}$



$\sigma\big(T; L^5(D) \oplus L^5(N)\big)$ when $\theta = \pi, 12\pi/10, 14\pi/10,\ 16\pi/10$



Lamé

For the Lamé operator we have:

Theorem (1)

For Ω infinite sector of aperture $\theta \in (0, 2\pi)$, $p \in (1, +\infty)$, and for $\boxed{\mathcal{L} = \mu \triangle \vec{u} + (\lambda + \mu)\nabla \mathrm{div}\, \vec{u}}$*, the operator T fails to be invertible whenever there exists $\xi \in \mathbb{R}$ such that:*

$$\boxed{\begin{aligned}&\left(\frac{\mu+\lambda}{3\mu+\lambda} \cdot \sin(\theta) \cdot (\frac{1}{p} + i\xi - 1)\right)^4 - \left(\frac{\mu+\lambda}{3\mu+\lambda} \cdot \sin(\theta) \cdot (\frac{1}{p} + i\xi - 1)\right)^2 \\ &\quad + \left(\sin(\frac{2\pi}{p} + 2\pi i\xi) + \sin(\frac{2\pi}{p} + 2\pi i\xi + \theta(1 - \frac{1}{p} - i\xi))\right)^2 \quad = 0\end{aligned}}$$

Note that when $\mu + \lambda = 0$ this is reduces to $\left(\sin(\frac{2\pi}{p} + 2\pi i\xi) + \sin(\frac{2\pi}{p} + 2\pi i\xi + \theta(1 - \frac{1}{p} - i\xi))\right) = 0$. This is the case $\mathcal{L} = \Delta$.

Lamé

Moreover, we were able to prove:

Theorem (2)

If $\theta \in (0, 2\pi - \frac{1}{400}]$ *and* $|\xi| > 8000$, *then:*

$$\boxed{\begin{aligned}&\left(\frac{\mu+\lambda}{3\mu+\lambda}\cdot\sin(\theta)\cdot(\tfrac{1}{p}+i\xi-1)\right)^4-\left(\frac{\mu+\lambda}{3\mu+\lambda}\cdot\sin(\theta)\cdot(\tfrac{1}{p}+i\xi-1)\right)^2\\&\qquad+\left(\sin(\tfrac{2\pi}{p}+2\pi i\xi)+\sin(\tfrac{2\pi}{p}+2\pi i\xi+\theta(1-\tfrac{1}{p}-i\xi))\right)^2\neq 0\end{aligned}}$$

for each $p \in (1, +\infty)$

Lamé

Theorem (3)

If $\theta \in (2\pi - \frac{1}{399}, 2\pi)$ *and* $|\xi| > \frac{40}{2\pi - \theta}$*, then:*

$$\left(\frac{\mu+\lambda}{3\mu+\lambda} \cdot \sin(\theta) \cdot (\frac{1}{p} + i\xi - 1)\right)^4 - \left(\frac{\mu+\lambda}{3\mu+\lambda} \cdot \sin(\theta) \cdot (\frac{1}{p} + i\xi - 1)\right)^2 + \left(\sin(\frac{2\pi}{p} + 2\pi i\xi) + \sin(\frac{2\pi}{p} + 2\pi i\xi + \theta(1 - \frac{1}{p} - i\xi))\right)^2 \neq 0$$

for each $p \in (1, +\infty)$

Conjectures

Conjecture (1)

For $p \in (1, +\infty)$ and $\theta \in (0, 2\pi)$ then:

$$\left(\tfrac{\mu+\lambda}{3\mu+\lambda} \cdot \sin(\theta) \cdot (\tfrac{1}{p} + i\xi - 1)\right)^4 - \left(\tfrac{\mu+\lambda}{3\mu+\lambda} \cdot \sin(\theta) \cdot (\tfrac{1}{p} + i\xi - 1)\right)^2 + \left(\sin(\tfrac{2\pi}{p} + 2\pi i\xi) + \sin(\tfrac{2\pi}{p} + 2\pi i\xi + \theta(1 - \tfrac{1}{p} - i\xi))\right)^2 = 0$$

$\implies$ $\boxed{\xi = 0}$.

Remarks:

- Conjecture OK whenever $\mu + \lambda = 0$.
- Given $\theta \in (0, 2\pi)$ – enough to consider ξ in a closed interval of $\mathbb{R}$.
- Thus methods of validated numerics can be relevant to this analysis. Currently pursuing this in collaboration with I. Mitrea & W. Tucker.

THANK YOU