Zero Verification: Interval-based Implementation of a Topological Test

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Uppsala, September 2016

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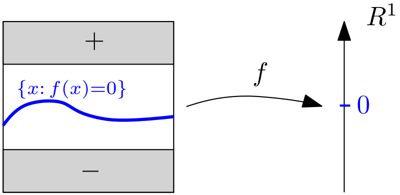

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- $m \geq n$
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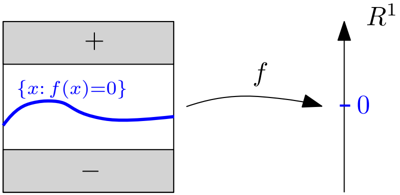
- test based on signs of f_j outside a neighborhood of 0
- robust wrt. perturbations of f



Extendability test

Observation

Let A be a region where $f(x) = 0$ has no solution. If there is no extension

$$\begin{array}{ccc} X & \dashrightarrow & \\ \cup\vert & & \\ A & \xrightarrow{f} & R^n \setminus \{0\}, \end{array}$$

then f has a zero in X.

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Idea of obstruction theory:

- we try to extend f to $X^{(k)}$ (inductively from small k)
- If we cannot extend to $X^{(k+1)}$, we cannot extent to all of X.

Primary obstruction

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Lemma

Let $f : X \to \mathbb{R}^n$, $A \subseteq X$ and f has no zero on A. Then f can always be extended to $X^{(n-1)}$ but not necessarily to $X^{(n)}$.

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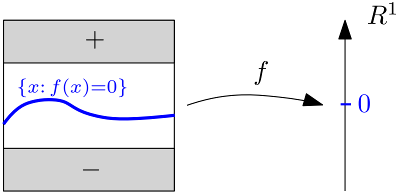

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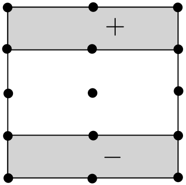

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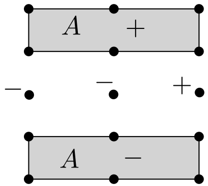

$n = 1$
f extendable to
0-skeleton of $[0, 1]^2$

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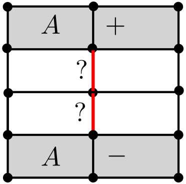

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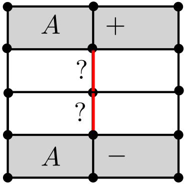

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Non-extendability to $X^{(n)}$ is measured by the primary obstruction (element of $H^n(X, A)$)

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- This work is an "interval" adaptation of the primary obstruction.
- "Regular" case $m = n$ described in [F, Ratschan, Zgliczynski. Quasi-decidability of ..., 2015]
- An explicit description of the algorithm follows.

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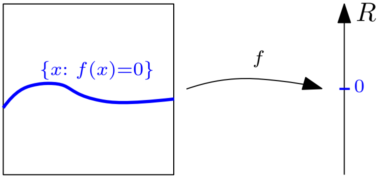
A consists of those boxes, where some f_k has constant sign.

A and sign information



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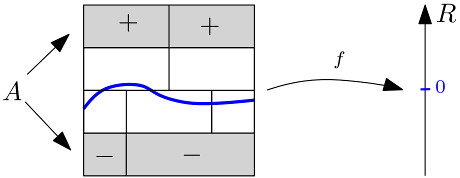

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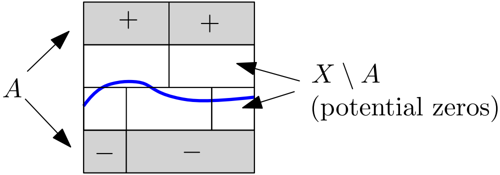

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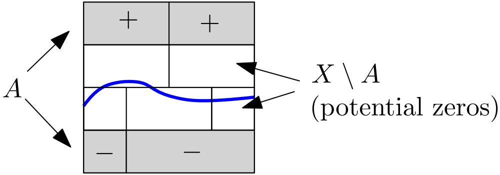

Lemma

Let A be a union of boxes and each box B of A is endowed with a pair (i, s), $1 \leq i \leq n$, $s \in \{+, -\}$ indicating that $f_i|_B$ has sign s.

Then the extendability of $f|_A$ to a map $X \to \mathbb{R}^n \setminus \{0\}$ is determined by the collection $\{B, (i_B, s_B)\}_{B \in A}$.

Labeling A-boxes with integers

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To each $(n-1)$-box B in A, we assign an integer

$$\xi(B) := \begin{cases} \deg(f_{\neg 1}, B) & \text{if } f_1 \text{ has sign } + \text{ on } B \\ 0 & \text{otherwise} \end{cases}$$

where $\deg$ is the degree and $f_{\neg 1}$ is f with the first component omitted. The degree is computable, given the "sign-information".

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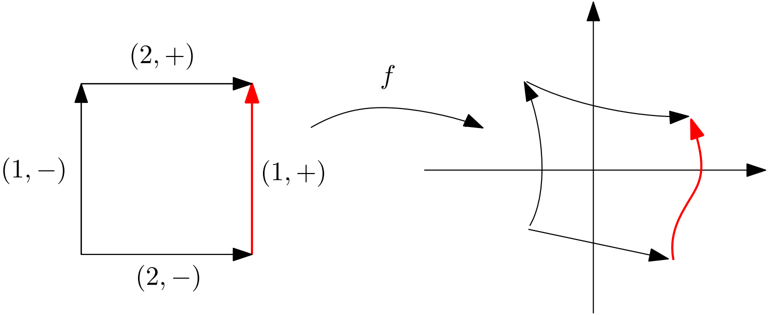

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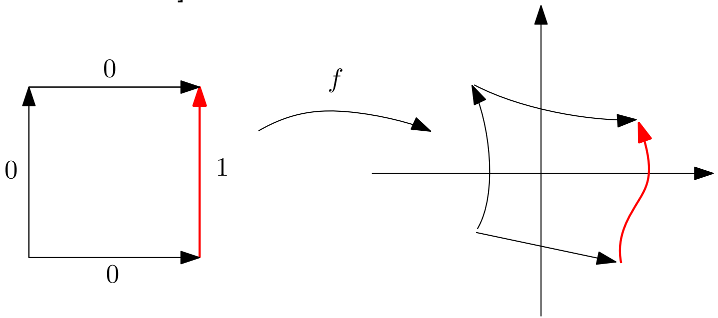

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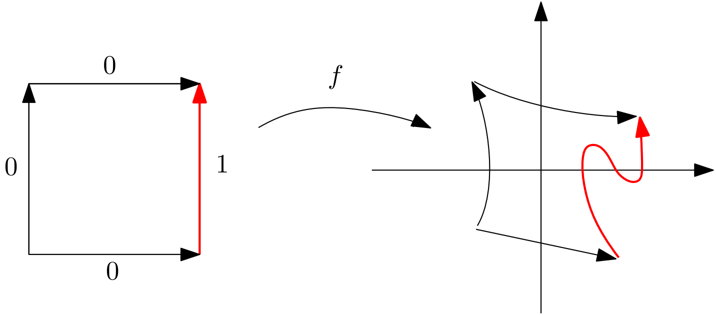

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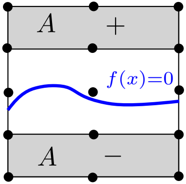

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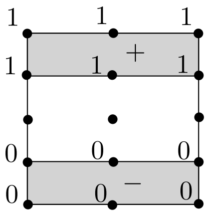

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Theorem

There exists an extension $X^{(n)} \to \mathbb{R}^n \setminus \{0\}$ *of* $f|_A$ *iff*

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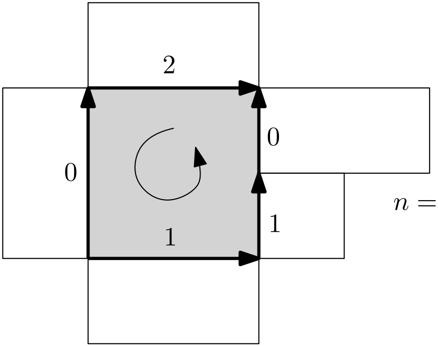

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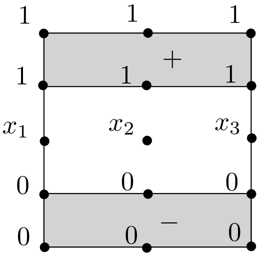

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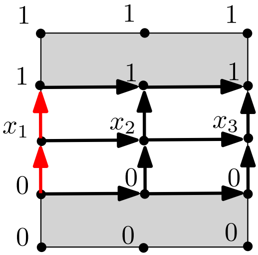

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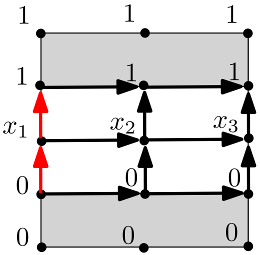

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No solution $\Rightarrow f|_A$ non-extendable $\Rightarrow f$ has a zero.

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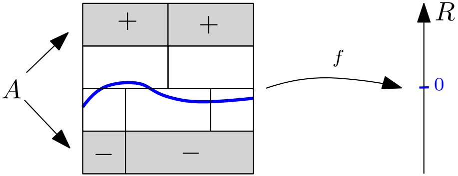
- Bottleneck: creation of A and collecting sign information (probably huge in high dimensions)



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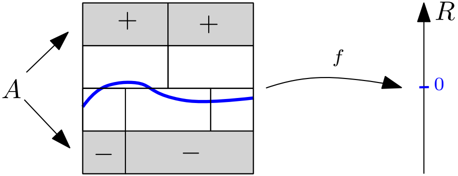

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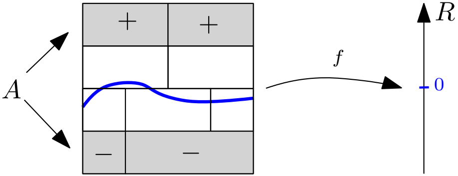

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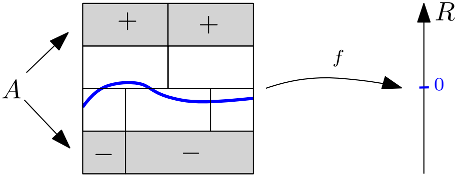

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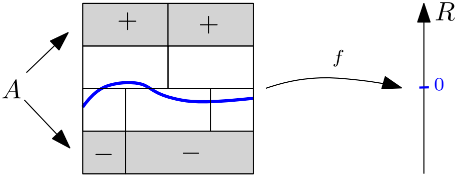

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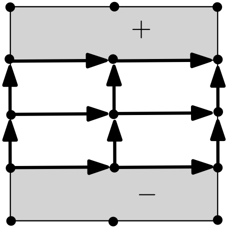
- Suitable for proving "robust" zero in situation when we don't care much where *exactly* the zero set is
- Computing higher obstruction in IA possible, but hard
- The algorithm may not verify anything

One possible simplification

The algorithm can likely be reduced: in this situation



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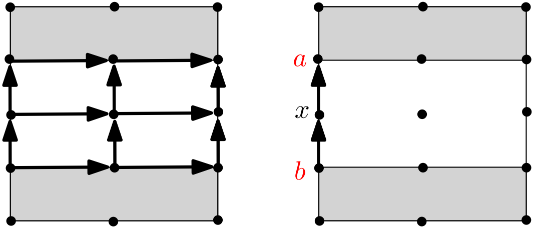

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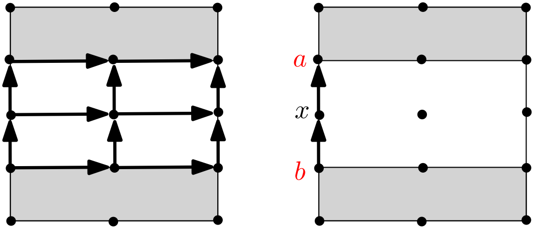

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Conjecture

We only need to consider n-cubes that support a set of generating cycle for the homology $H_n(X, A)$.

(Many software packages can compute these generators)

Approximating robustness

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If $r \leq |f(x)| \leq R$ *for all* $x \in A$*, then*

- *non-extendability of* $f|_A$ *implies that every function* g*,* $\|g - f\|_\infty \leq r$*, has a zero in* X*,*
- *extendability of* $f|_A$ *implies that some function* g*,* $\|g - f\|_\infty \leq R$ *has no zero in* X*.*

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This could be applied for measuring "robustness" of the zero set.

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Thank you for your attention.