Decision Making Under Interval Uncertainty as a Natural Example of a Quandle

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1. Outline

- In many real-life situations, we need to select an alternative from a set of possible alternatives.
- In many such situations, we have a well-defined objective function $u(a)$ that describes our preferences.
- If we know the exact value of $u(a)$ for each alternative a, then we select the alternative with the largest $u(a)$.
- In practice, however, we usually know the consequences of each decision a only with some uncertainty.
- As a result, for each alternative a, we only know the interval of possible values $[\underline{u}(a), \overline{u}(a)]$.
- We show: decision making under interval uncertainty is an example of a *quandle* – a knot-theory concept.

2. Need for Decision Making

- In many real-life situations, we need to select an alternative a from the list of possible alternatives; e.g.:
  - we want to select a design and/or location of a plant,
  - we want to select a financial investment, etc.
- In many such situations, we have a well-defined objective function $u(a)$ that describes our preferences:
  - if we know the exact value of $u(a)$ for each alternative a,
  - then we select the alternative with the largest value of $u(a)$.

3. Decision Making Under Interval Uncertainty

- In practice, we usually only know the consequences of each decision with some uncertainty; often:
  - the only information that we have about the corresponding values of $u(a)$
  - is that $u(a)$ is somewhere between the known bounds $\underline{u}(a)$ and $\overline{u}(a)$,
  - i.e., that $u(a) \in [\underline{u}(a), \overline{u}(a)]$.
- How can we make a decision under such interval uncertainty?

4. To Make a Decision under Interval Uncertainty, We Need to Select a Value from the Interval

- To make a decision under interval uncertainty, we need, in particular, to be able to compare:
  - the alternative a for which we only know the interval of possible values of $u(a)$,
  - with alternatives b for which we know the exact utility values $u(b)$.
- For some values $u(b)$, the alternative b is better; for others, a is better.
- Clearly, if a is better than b and $u(b) > u(c)$, then a should be better than c as well.
- Similarly, if a is worse than b and $u(b) < u(c)$, then a should be worse than c as well.

5. Selecting a Value from the Interval (cont-d)

- Thus, there should be a threshold value u_0 that separates:
  - alternatives b for which a is better
  - from alternatives b' for which a is better.
- In other words, when we make decisions, we compare $u(b)$ with this threshold value u_0.
- This value u_0 thus represents the utility of the alternative for which we only know the interval $[\underline{u}(a), \overline{u}(a)]$.
- We therefore need to be able, given an interval $[\underline{u}(a), \overline{u}(a)]$, to produce an equivalent utility value u_0.
- Let us denote this value u_0 by $\overline{u}(a) \rhd \underline{u}(a)$.
- *Main problem:* which operation $\rhd$ should we select?

6. Natural Properties of $\rhd$

- In order to answer the above questions, let us analyze what are the natural properties of the operation $a \rhd b$.
- If we know the exact value of $u(a)$, then the equivalent value is simply equal to x: $x \rhd x = x$.
- Another reasonable property is *monotonicity:* if $x < x'$, then $x \rhd y > x' \rhd y$.
- Small changes in x and y should lead to small changes in the equivalent value $x \rhd y$.
- In other words, the operation $\rhd$ should be *continuous.*

7. Twin Interval Uncertainty

- In practice:
  - instead of knowing the exact bounds $\underline{u}(a)$ and $\overline{u}(a)$ on $u(a)$,
  - we may only know the bounds on each of these bounds:

$$\underline{u}(a) \in [\underline{u}^-(a), \underline{u}^+(a)] \text{ and } \overline{u}(a) \in [\overline{u}^-(a), \overline{u}^+(a)].$$

- Such a situation is known as *twin interval uncertainty.*
- For example:
  - we may know the lower bound z of the corresponding interval,
  - but we only know that the upper bound is between y and x.
- We can analyze this situations in two different ways.

8. Twin Interval Uncertainty (cont-d)

- First, all we know about the upper bound is that it is between y and x.
- Thus, this upper bound is therefore equivalent to the value $y \rhd x$.
- Now, we have an interval with an exact lower bound z and an exact upper bound $x \rhd y$.
- We can now apply the operation $\rhd$ to estimate the equivalent value of this interval as $(x \rhd y) \rhd z$.
- There is also an alternative approach.
- For each possible value v between y and x, we have an interval $[z, v]$ with equivalent value $v \rhd z$.

9. Twin Interval Uncertainty (cont-d)

- Due to monotonicity, the equivalent value $v \rhd z$:
  - is the smallest when v is the smallest, i.e., when $v = y$, and
  - it is the largest when v is the largest, i.e., when $v = x$.
- These equivalent values form an interval $[y \rhd z, x \rhd z]$.
- The equivalent value of this interval is $(x \rhd z) \rhd (y \rhd z)$.
- It is reasonable to require that these two approaches lead to the same value: $(x \rhd y) \rhd z = (x \rhd z) \rhd (y \rhd z)$.
- Similarly, we can consider situations in which:
  - we know the upper bound x of the interval, but
  - we only know the lower bound is between y and z.
- In this case, a similar analysis leads to the requirement that $x \rhd (y \rhd z) = (x \rhd y) \rhd (x \rhd z)$.

10. This Is A Quandle

- If we know that $u \in [\underline{u}, \overline{u}]$, then $\underline{u} \le u \le \overline{u}$, so we should have $u_0 \in [\underline{u}, \overline{u}]$.
- Interestingly, the above properties (plus an appropriate monotonicity) are well known in knot theory:

$$x \triangleright x = x, \quad (x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z),$$

$$(y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z).$$

- Sets with operations satisfying these properties are knows as *quandles*.
- Let us use this relation to describe possible operations $\triangleright$ for decision making under interval uncertainty.

11. Discussion

- In general, the operation $\triangleright$ is monotonically increasing with respect to each of its variables.
- For differentiable functions, this implies that both partial derivatives are non-negative.
- Our result, however, requires a stronger condition: that both derivatives are always positive.
- We also need to require:
    - not only that $x \triangleright y \in [y, x]$,
    - but also that $x \triangleright y \in (y, x)$ for $y < x$,
    - i.e., that the degenerate cases $x \triangleright y = x$ and $x \triangleright y = y$ are excluded.
- Under these conditions, we prove the following result.

12. Main Result

- We say that a differentiable function $f(x_1, \ldots, x_m)$ is *strongly increasing* if all its partial derivatives are > 0.
- Let $x \rhd y$ be a continuously differentiable strongly increasing function defined for all $x \geq y$ for which:

$$x \rhd x = x, \quad (x \rhd y) \rhd z = (x \rhd z) \rhd (y \rhd z),$$

$$(y \rhd z) = (x \rhd y) \rhd (x \rhd z), \quad x \rhd y \in (y, x) \text{ when } y < x.$$

- Then, for some continuous strictly increasing function $f(x)$ and for some $\alpha \in (0, 1)$:

$$x \rhd y = f^{-1}(\alpha \cdot f(x) + (1 - \alpha) \cdot f(y))$$

13. Discussion

- In other words, after an appropriate monotonic re-scaling $x \to X = f(x)$, we get $X \rhd Y = \alpha \cdot X + (1 - \alpha) \cdot Y$.
- This way of making decisions under interval uncertainty is well known.
- This combination of best-case and worst-case was originally proposed by the Nobelist Leo Hurwicz.
- Our result provides a new justification for Hurwicz's criterion.
- *Caution:* by requiring that $x \rhd y \in (y, x)$ when $y < x$, we exclude the following two extreme cases:
  - the super-optimistic case $\alpha = 1$, when the decision maker only takes into account the best case; and
  - the super-pessimistic case $\alpha = 0$, when the decision maker only takes into account the worst case.

14. Open Questions

- What if we only require that $x \rhd y \in [y, x]$?
- What if we only require monotonicity – and allow zero values of the derivatives?
- What if we only require continuity instead of differentiability?

15. Need for Improper Intervals

- It is sometimes useful to consider *improper* intervals $[a, b]$, with $a > b$.
- Let us consider the case when a decision maker is participating in two different situations:
  - in the first situation, the decision maker gains some amount $u \in [\underline{u}, \overline{u}]$;
  - in the second situation, the decision maker gains some amount $v \in [\underline{v}, \overline{v}]$.
- The possible values of the amount $u + v$ gained by the decision maker form an interval $[\underline{u} + \underline{v}, \overline{v} + \overline{v}]$.
- Suppose that:
  - after decision makers gain $u \in [\underline{u}, \overline{u}]$,
  - we want to compensate them, so that each of them gains $\underline{u} + \overline{u}$ (double the average gain).

16. Need for Improper Intervals (cont-d)

- Possible values of the compensation v form the interval $[\underline{u}, \overline{u}]$.
- However, we want to avoid the conclusion that $u + v \in [\underline{u} + \underline{u}, \overline{u} + \overline{u}]$.
- We thus say that the possible values of the compensation amount v form an *improper* interval $[\overline{u}, \underline{u}]$.
- In this case, $u \in [\underline{u}, \overline{u}]$ and $v \in [\overline{u}, \underline{u}]$ imply that

$$u + v \in [\underline{u} + \overline{u}, \overline{u} + \underline{u}].$$

- So, we conclude – correctly this time – that the overall compensation is always equal to $\underline{u} + \overline{u}$.
- It is reasonable to extent the question of selecting an appropriate value u_0 to such improper intervals as well.
- In this case, the operation $x \rhd y$ is defined for all possible pairs of real numbers (x, y).

17. Results

- If we allow improper intervals,
  - then we can relax some of the restrictions that we placed on the operation $\rhd$ in Proposition 1
  - but for that, we need to require that *both* conditions be satisfied:

$$(x \rhd y) \rhd z = (x \rhd z) \rhd (y \rhd z), \quad (y \rhd z) = (x \rhd y) \rhd (x \rhd z).$$

- If a function $x \rhd y$ is continuous, strictly increasing, and satisfies both conditions, then

$$x \rhd y = f^{-1}(\alpha \cdot f(x) + (1 - \alpha) \cdot f(y)).$$

- If a function $x \rhd y$ is differentiable, strictly increasing, and satisfies one of the conditions, then

$$x \rhd y = f^{-1}(\alpha \cdot f(x) + (1 - \alpha) \cdot f(y)).$$

18. Results (cont-d)

- If a function $x \rhd y$ is continuous, strictly increasing,
- satisfies one of the conditions

$$(x \rhd y) \rhd z = (x \rhd z) \rhd (y \rhd z), \quad (y \rhd z) = (x \rhd y) \rhd (x \rhd z),$$

- and for every x and y, there exist z' and z'' for which $x \rhd z' = z'' \rhd x = y$, then

$$x \rhd y = f^{-1}(\alpha \cdot f(x) + (1 - \alpha) \cdot f(y)).$$

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