Fast validated computation for solutions of algebraic Riccati equations arising in transport theory

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Nonsymmetric algebraic Riccati equations arising in transport theory

Find $X \in \mathbb{R}^{n \times n}$ s.t. $XCX - XE - AX + B = 0$, where

$A = \Delta - eq^T$, $B = ee^T$, $C = qq^T$, $E = D - qe^T$, $e = (1, \ldots, 1)^T$, $q = (q_1, \ldots, q_n)^T$, $q_i = c_i/(2\omega_i)$, $\Delta = \text{diag}(\delta_1, \ldots, \delta_n)$,

$\delta_i = 1/(c\omega_i(1 + \alpha))$, $D = \text{diag}(d_1, \ldots, d_n)$, $d_i = 1/(c\omega_i(1 - \alpha))$,

$0 < c \leq 1$, $0 \leq \alpha < 1$, $0 < \omega_n < \cdots < \omega_1 < 1$, $\sum_{i=1}^{n} c_i = 1$, $c_i > 0$

Sol. of interest: minimal positive sol. (componentwise sense)
Motivation

Efficient numerical algorithms (at least 21 publications)
⇒ They cannot provide an exact solution.
⇒ We want to give reliability to the obtained approximations.

Purpose

Numerically computing $X^r$ s.t. $|\tilde{X} - X| \leq X^r$, where

$\tilde{X} = \text{numerical sol., } |M| = (|M_{ij}|), M \leq N \Leftrightarrow M_{ij} \leq M_{ij}, \forall i, j.$

⇒ If $X_{ij}^r$ is small, $\tilde{X}_{ij}$ is reliable. Moreover, $X \in \langle \tilde{X}, X^r \rangle$. 
Previous work

No publications for nonsymmetric algebraic Riccati equations

How to reduce the cost exploiting the special structure?

Our contribution: verification algorithm is proposed

- $O(n^2)$ ops. under a reasonable assumption
- The local uniqueness and minimal positiveness are also verified.
Vector equations

$X = T \circ (uv^T)$, where $T = (1/(\delta_i + d_j))$, $u > 0$ and $v > 0$ are sol.

of

$$\begin{cases} u - u \circ (Pv) - e = 0 \\
v - v \circ (Qu) - e = 0 \end{cases}$$

$P = T \text{diag}(q), \quad Q = T^T \text{diag}(q)$

$\Rightarrow$ We compute $u$ and $v$ s.t. $u \ni u$ and $v \ni v. \Rightarrow X \in T \circ (uv^T)$

How to compute $u$ and $v$? (1/3)

Vector equations $\Leftrightarrow f(w) = 0$, where $w = (u^T, v^T)^T$,

$$f(w) = \begin{pmatrix} u - u \circ (Pv) - e \\ v - v \circ (Qu) - e \end{pmatrix} = w - w \circ (Rw) - e, \quad R = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}$$
How to compute $u$ and $v$? (2/3)

$J(w)$: the Jacobian matrix of $f(w)$ (Z-matrix if $w > 0$)

$J(w) = D(w) - L(w) = I - K(w)$, where $D(w) = I - \text{diag}(Rw)$,

$L(w) = \text{diag}(w)R$, $K(w) = \text{diag}(Rw) + L(w)$

If $J(\tilde{w})$ is nonsingular, we can define $n(w) = w - J(\tilde{w})^{-1}f(w)$

($\tilde{w}$: approx. sol. of $f(w) = 0$) Note that $f(w) = 0 \Leftrightarrow w = n(w)$.

$\Rightarrow$ We verify the nonsingularity of $J(\tilde{w})$ and

$\{n(w) : w \in \langle \tilde{w}, w^r \rangle \} \subseteq \text{int}(\langle \tilde{w}, w^r \rangle)$ for given $w^r > 0$. 
How to compute $u$ and $v$? (3/3)

If they are true, $w^* \in \text{int}(\langle \tilde{w}, w^r \rangle)$, where $w^* = n(w^*) (f(w^*) = 0)$.

$$w^* = n(w^*) \in \{n(w) : w \in \langle \tilde{w}, w^r \rangle\}$$

$$\Rightarrow w = (u^T, v^T)^T$$

can be computed s.t. $\{n(w) : w \in \langle \tilde{w}, w^r \rangle\} \subseteq w$

How to verify the nonsingularity?

If $\exists x \in \mathbb{R}^{2n}$ s.t. $x > 0$ and $J(\tilde{w})x > 0$, then $J(\tilde{w})$ is an $M$-matrix.

$\Rightarrow$ We prove the nonsingularity by computing such $x$. 
Verifying \( \{n(w) : w \in \langle \tilde{w}, w^r \rangle \} \subseteq \text{int}(\langle \tilde{w}, w^r \rangle) \)

We compute \( w^\varepsilon \) s.t. \( \{n(w) : w \in \langle \tilde{w}, w^r \rangle \} \subseteq \langle \tilde{w}, w^\varepsilon \rangle \) and check \( w^\varepsilon < w^r \). We can obtain \( w^\varepsilon \) by the following idea:

\[
 n(w) = w - J(\tilde{w})^{-1}f(w) \iff J(\tilde{w})n(w) = J(\tilde{w})w - f(w)
\]

\( \Rightarrow \{n(w) : w \in \langle \tilde{w}, w^r \rangle \} \) is the set of all sol. of linear systems

\[
 J(\tilde{w})n_w = J(\tilde{w})w - f(w),
\]

where \( n_w \in \mathbb{R}^{2n} \) is unknown and \( w \in \langle \tilde{w}, w^r \rangle \) is the parameter.

The sol. set can be enclosed with \( \mathcal{O}(n^2) \) ops. by reusing \( x \).
How to compute $x$ s.t. $x > 0$ and $J(\tilde{w})x > 0$?

- $J(\tilde{w}) = D(\tilde{w}) - L(\tilde{w})$ is an $M$-matrix $\iff \rho(D(\tilde{w})^{-1}L(\tilde{w})) < 1$
- $D(\tilde{w})^{-1}L(\tilde{w})$ is nonnegative and irreducible.

$\Rightarrow D(\tilde{w})^{-1}L(\tilde{w})$ has eigenvalue $\rho(D(\tilde{w})^{-1}L(\tilde{w}))$ and corresponding positive eigenvector.

$\Rightarrow x^{(k+1)} = D(\tilde{w})^{-1}L(\tilde{w})x^{(k)}$, $k = 0, 1, \ldots$ for $x^{(0)} > 0$ implies

$\lambda^{(k)} = \max_i x_i^{(k+1)}/x_i^{(k)} \rightarrow \rho(D(\tilde{w})^{-1}L(\tilde{w}))$ (monotonic decrease)

**Theorem 1**

$J(\tilde{w})x^{(k)} > 0 \iff \lambda^{(k)} < 1$

**Theorem 2**

If $0 < \tilde{w} \leq w^* < 2e$ and $x^{(0)} = \tilde{w}$, then $\lambda^{(0)} < 1$. 
Alternative iteration (1/2)

\[ S(\tilde{w}) = S_Q(\tilde{w})S_P(\tilde{w}), \text{ where } S_P(\tilde{w}) = (I - \text{diag}(P\tilde{v}))^{-1}\text{diag}(\tilde{u})P, \]
\[ S_Q(\tilde{w}) = (I - \text{diag}(Q\tilde{u}))^{-1}\text{diag}(\tilde{v})Q \]

Then, \( \det(D(\tilde{w})^{-1}L(\tilde{w}) - \lambda I) = 0 \iff \det(S(\tilde{w}) - \lambda^2 I) = 0 \)

Therefore, \( \rho(D(\tilde{w})^{-1}L(\tilde{w})) < 1 \iff \rho(S(\tilde{w})) < 1 \)

\( S(\tilde{w}) > 0 \implies S(\tilde{w}) \) has eigenvalue \( \rho(S(\tilde{w})) \) and corresponding positive eigenvector.

\( \implies p^{(k+1)} = S(\tilde{w})p^{(k)}, \, k = 0, 1, \ldots \text{ for } p^{(0)} > 0 \text{ implies } \)

\[ \mu^{(k)} = \max_i p^{(k)}_i / p^{(k)}_i \to \rho(S(\tilde{w})) \text{ (monotonic decrease)} \]
Alternative iteration (2/2)

**Theorem 3** Let \( \hat{x}^{(k)} = ((S_P(\tilde{w})p^{(k)})^T, \sqrt{\mu^{(k)}p^{(k)}T})^T \). Then, \( J(\tilde{w})\hat{x}^{(k)} > 0 \iff \mu^{(k)} < 1 \).

**Theorem 4** If \( 0 < \tilde{w} \leq w^* < 2e \) and \( p^{(0)} = \tilde{v} \circ (v^* - e) \), then \( \mu^{(0)} < 1 \) \((\iff p^{(0)} = \tilde{v} \circ (\tilde{v} - e) \) is a good choice).

**Convergence analysis for 1st iteration**

Eigenvalues of \( D(\tilde{w})^{-1}L(\tilde{w}) \): \( \lambda_1 = |\lambda_2| > |\lambda_3| \geq \cdots \geq |\lambda_{2n}| \)

Convergence rate of \( x^{(k+1)} = D(\tilde{w})^{-1}L(\tilde{w})x^{(k)} \): \( |\lambda_3|/\lambda_1 \)
Convergence analysis for 2nd iteration

eigenvalues of $S(\tilde{w})$: $\lambda_1^2 > |\lambda_3|^2 \geq \cdots \geq |\lambda_{2n-1}|^2$

Convergence rate of $p^{(k+1)} = S(\tilde{w})p^{(k)}$: $(|\lambda_3|/\lambda_1)^2$

Comparison by numerical examples

Intel Core 2.60GHz CPU, 8.00GB RAM, MATLAB R2012a with Intel MKL and IEEE 754 double precision

c_i, \omega_i$: a numerical quadrature formula on the interval $[0, 1]$, obtained by dividing $[0, 1]$ into $n/4$ subintervals of equal length and applying a Gauss-Legendre quadrature with 4 nodes to each subinterval
Comparison for various $n$

$\tilde{w}$: N. Huang, C.F. Ma, Some predictor-corrector-type iterative schemes for solving nonsymmetric algebraic Riccati equations arising in transport theory, Numer. Linear Algebra Appl. 21, 761–780 (2014)

$$(\alpha, c) = (10^{-1}, 1 - 10^{-1})$$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\max \tilde{w}_i$</th>
<th>$D(\tilde{w})^{-1}L(\tilde{w})$</th>
<th>$S(\tilde{w})$</th>
<th>CPU times (sec)</th>
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<td>2048</td>
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Comparison for various \((\alpha, c)\)

\(n = 128\)

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<th>((\alpha, c))</th>
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<th>numbers of iterations</th>
<th>CPU times (sec)</th>
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Local uniqueness and minimal positiveness

**Theorem 5** (Juang (1995)) \( X > 0 \) s.t.

\[
\begin{align*}
(c(1+\alpha)/2)\gamma^T(Xq+e) &\leq 1 \\
(c(1-\alpha)/2)\gamma^T(X^Tq+e) &\leq 1
\end{align*}
\]

is (globally) unique and minimal positive.

\( \Rightarrow \) We check 2 inequalities with the enclosure of \( X \).
Numerical results

$M$: proposed algorithm ($\mathcal{O}(n^2)$ ops. if $\mathcal{O}(1)$ iterations)

$V$: verifynlss for $f(w) = 0$ ($\mathcal{O}(n^3)$ ops.)

maximum radius $= \max_{i,j} X_{i,j}$, where $\langle \tilde{X}, X^r \rangle \ni X$

time ratio $= \frac{\text{CPU time of } M \text{ or } V}{\text{CPU time for computing } \tilde{w}}$

$M$ and $V$ includes the computation of $\tilde{w} \Rightarrow$ time ratio $> 1$

If time ratio $< 2$, verification is faster than the computation of $\tilde{w}$. 
Results for various $n$

$$(\alpha, c) = (10^{-1}, 1 - 10^{-1})$$

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<th>$n$</th>
<th>maximum radii</th>
<th>time ratios</th>
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\[ n = 128 \]

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<td>V</td>
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