Fast validated computation for solutions of algebraic Riccati equations arising in transport theory

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Nonsymmetric algebraic Riccati equations arising in transport theory

Find $X \in \mathbb{R}^{n\times n}$ s.t. $XCX - XE - AX + B = 0$, where

$A = \Delta - eq^T$, $B = ee^T$, $C = qq^T$, $E = D - qe^T$, $e = (1,\dots,1)^T$,

$q = (q_1,\dots,q_n)^T$, $q_i = c_i/(2\omega_i)$, $\Delta = \mathrm{diag}(\delta_1,\dots,\delta_n)$,

$\delta_i = 1/(c\omega_i(1+\alpha))$, $D = \mathrm{diag}(d_1,\dots,d_n)$, $d_i = 1/(c\omega_i(1-\alpha))$,

$0 < c \le 1$, $0 \le \alpha < 1$, $0 < \omega_n < \cdots < \omega_1 < 1$, $\sum_{i=1}^n c_i = 1$, $c_i > 0$

Sol. of interest: minimal positive sol. (componentwise sense)

Motivation

Efficient numerical algorithms (at least 21 publications)

$\Rightarrow$ They cannot provide an exact solution.

$\Rightarrow$ We want to give reliability to the obtained approximations.

Purpose

Numerically computing X^r s.t. $|\tilde{X} - X| \leq X^r$, where

$\tilde{X} =$ numerical sol., $|M| = (|M_{ij}|)$, $M \leq N \Leftrightarrow M_{ij} \leq M_{ij}, \forall i, j$.

$\Rightarrow$ If X^r_{ij} is small, $\tilde{X}_{ij}$ is reliable. Moreover, $X \in \langle \tilde{X}, X^r \rangle$.

Previous work

No publications for nonsymmetric algebraic Riccati equations

How to reduce the cost exploiting the special structure?

Our contribution: verification algorithm is proposed

- $\mathcal{O}(n^2)$ ops. under a reasonable assumption
- The local uniqueness and minimal positiveness are also verified.

Vector equations

$X = T \circ (uv^T)$, where $T = (1/(\delta_i + d_j))$, $u > 0$ and $v > 0$ are sol.

of $\begin{cases} u - u \circ (Pv) - e = 0 \\ v - v \circ (Qu) - e = 0 \end{cases}$, $P = T\mathrm{diag}(q)$, $Q = T^T\mathrm{diag}(q)$

$\Rightarrow$ We compute $\boldsymbol{u}$ and $\boldsymbol{v}$ s.t. $\boldsymbol{u} \ni u$ and $\boldsymbol{v} \ni v$. $\Rightarrow X \in T \circ (\boldsymbol{u}\boldsymbol{v}^T)$

How to compute $\boldsymbol{u}$ and $\boldsymbol{v}$? (1/3)

Vector equations $\Leftrightarrow f(w) = 0$, where $w = (u^T, v^T)^T$,

$$f(w) = \begin{pmatrix} u - u \circ (Pv) - e \\ v - v \circ (Qu) - e \end{pmatrix} = w - w \circ (Rw) - e, \ R = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}$$

How to compute u and v? (2/3)

$J(w)$: the Jacobian matrix of $f(w)$ (Z-matrix if $w > 0$)

$J(w) = D(w) - L(w) = I - K(w)$, where $D(w) = I - \mathrm{diag}(Rw)$,

$L(w) = \mathrm{diag}(w)R$, $K(w) = \mathrm{diag}(Rw) + L(w)$

If $J(\tilde{w})$ is nonsingular, we can define $n(w) = w - J(\tilde{w})^{-1}f(w)$

($\tilde{w}$: approx. sol. of $f(w) = 0$) Note that $f(w) = 0 \Leftrightarrow w = n(w)$.

$\Rightarrow$ We verify the nonsingularity of $J(\tilde{w})$ and

$\{n(w) : w \in \langle \tilde{w}, w^r \rangle\} \subseteq \mathrm{int}(\langle \tilde{w}, w^r \rangle)$ for given $w^r > 0$.

How to compute u and v? (3/3)

If they are true, $w^* \in \mathrm{int}(\langle \tilde{w}, w^r \rangle)$, where $w^* = n(w^*)$ ($f(w^*) = 0$).

$\Rightarrow w^* = n(w^*) \in \{n(w) : w \in \langle \tilde{w}, w^r \rangle\}$

$\Rightarrow \boldsymbol{w} = (\boldsymbol{u}^T, \boldsymbol{v}^T)^T$ can be computed s.t. $\{n(w) : w \in \langle \tilde{w}, w^r \rangle\} \subseteq \boldsymbol{w}$

How to verify the nonsingularity?

If $\exists x \in \mathbb{R}^{2n}$ s.t. $x > 0$ and $J(\tilde{w})x > 0$, then $J(\tilde{w})$ is an M-matrix.

$\Rightarrow$ We prove the nonsingularity by computing such x.

Verifying $\{n(w) : w \in \langle \tilde{w}, w^r \rangle\} \subseteq \mathrm{int}(\langle \tilde{w}, w^r \rangle)$

We compute w^ε s.t. $\{n(w) : w \in \langle \tilde{w}, w^r \rangle\} \subseteq \langle \tilde{w}, w^\varepsilon \rangle$ and check $w^\varepsilon < w^r$. We can obtain w^ε by the following idea:

$$n(w) = w - J(\tilde{w})^{-1} f(w) \Leftrightarrow J(\tilde{w}) n(w) = J(\tilde{w}) w - f(w)$$

$\Rightarrow \{n(w) : w \in \langle \tilde{w}, w^r \rangle\}$ is the set of all sol. of linear systems

$$J(\tilde{w}) n_w = J(\tilde{w}) w - f(w),$$

where $n_w \in \mathbb{R}^{2n}$ is unknown and $w \in \langle \tilde{w}, w^r \rangle$ is the parameter.

The sol. set can be enclosed with $\mathcal{O}(n^2)$ ops. by reusing x.

How to compute x s.t. $x > 0$ and $J(\tilde{w})x > 0$?

- $J(\tilde{w}) = D(\tilde{w}) - L(\tilde{w})$ is an M-matrix $\Leftrightarrow \rho(D(\tilde{w})^{-1}L(\tilde{w})) < 1$
- $D(\tilde{w})^{-1}L(\tilde{w})$ is nonnegative and irreducible.

$\Rightarrow D(\tilde{w})^{-1}L(\tilde{w})$ has eigenvalue $\rho(D(\tilde{w})^{-1}L(\tilde{w}))$ and corresponding positive eigenvector.

$\Rightarrow x^{(k+1)} = D(\tilde{w})^{-1}L(\tilde{w})x^{(k)}$, $k = 0, 1, \ldots$ for $x^{(0)} > 0$ implies

$\lambda^{(k)} = \max_i x_i^{(k+1)}/x_i^{(k)} \to \rho(D(\tilde{w})^{-1}L(\tilde{w}))$ (monotonic decrease)

Theorem 1 $J(\tilde{w})x^{(k)} > 0 \Leftrightarrow \lambda^{(k)} < 1$

Theorem 2 If $0 < \tilde{w} \leq w^* < 2e$ and $x^{(0)} = \tilde{w}$, then $\lambda^{(0)} < 1$.

Alternative iteration (1/2)

$S(\tilde{w}) = S_Q(\tilde{w})S_P(\tilde{w})$, where $S_P(\tilde{w}) = (I - \mathrm{diag}(P\tilde{v}))^{-1}\mathrm{diag}(\tilde{u})P$, $S_Q(\tilde{w}) = (I - \mathrm{diag}(Q\tilde{u}))^{-1}\mathrm{diag}(\tilde{v})Q$

Then, $\det(D(\tilde{w})^{-1}L(\tilde{w}) - \lambda I) = 0 \Leftrightarrow \det(S(\tilde{w}) - \lambda^2 I) = 0$

Therefore, $\rho(D(\tilde{w})^{-1}L(\tilde{w})) < 1 \Leftrightarrow \rho(S(\tilde{w})) < 1$

$S(\tilde{w}) > 0 \Rightarrow S(\tilde{w})$ has eigenvalue $\rho(S(\tilde{w}))$ and corresponding positive eigenvector.

$\Rightarrow p^{(k+1)} = S(\tilde{w})p^{(k)}$, $k = 0, 1, \ldots$ for $p^{(0)} > 0$ implies

$\mu^{(k)} = \max_i p_i^{(k+1)}/p_i^{(k)} \to \rho(S(\tilde{w}))$ (monotonic decrease)

Alternative iteration (2/2)

Theorem 3 Let $\hat{x}^{(k)} = ((S_P(\tilde{w})p^{(k)})^T, \sqrt{\mu^{(k)}}p^{(k)^T})^T$. Then, $J(\tilde{w})\hat{x}^{(k)} > 0 \Leftrightarrow \mu^{(k)} < 1$.

Theorem 4 If $0 < \tilde{w} \leq w^* < 2e$ and $p^{(0)} = \tilde{v} \circ (v^* - e)$, then $\mu^{(0)} < 1$ ($\Rightarrow p^{(0)} = \tilde{v} \circ (\tilde{v} - e)$ is a good choice).

Convergence analysis for 1st iteration

eigenvalues of $D(\tilde{w})^{-1}L(\tilde{w})$: $\lambda_1 = |\lambda_2| > |\lambda_3| \geq \cdots \geq |\lambda_{2n}|$

Convergence rate of $x^{(k+1)} = D(\tilde{w})^{-1}L(\tilde{w})x^{(k)}$: $|\lambda_3|/\lambda_1$

Convergence analysis for 2nd iteration

eigenvalues of $S(\tilde{w})$: $\lambda_1^2 > |\lambda_3|^2 \geq \cdots \geq |\lambda_{2n-1}|^2$

Convergence rate of $p^{(k+1)} = S(\tilde{w})p^{(k)}$: $(|\lambda_3|/\lambda_1)^2$

Comparison by numerical examples

Intel Core 2.60GHz CPU, 8.00GB RAM, MATLAB R2012a with Intel MKL and IEEE 754 double precision

c_i, ω_i: a numerical quadrature formula on the interval $[0, 1]$, obtained by dividing $[0, 1]$ into $n/4$ subintervals of equal length and applying a Gauss-Legendre quadrature with 4 nodes to each subinterval

Comparison for various n

$\tilde{w}$: N. Huang, C.F. Ma, Some predictor-corrector-type iterative schemes for solving nonsymmetric algebraic Riccati equations arising in transport theory, Numer. Linear Algebra Appl. 21, 761–780 (2014)

$(\alpha, c) = (10^{-1}, 1 - 10^{-1})$

		numbers of iterations		CPU times (sec)	
n	$\max \tilde{w}_i$	$D(\tilde{w})^{-1}L(\tilde{w})$	$S(\tilde{w})$	$D(\tilde{w})^{-1}L(\tilde{w})$	$S(\tilde{w})$
256	1.88	1	1	4.8e–3	4.8e–3
512	1.88	1	1	1.1e–2	1.2e–2
1024	1.88	1	1	4.0e–2	3.7e–2
2048	1.88	1	1	1.4e–1	1.4e–1
4096	1.88	1	1	5.3e–1	5.3e–1
8192	1.88	1	1	2.1e+0	2.1e+0

Comparison for various (α, c)

$n = 128$

(α, c)	$\max \tilde{w}_i$	numbers of iterations $D(\tilde{w})^{-1}L(\tilde{w})$	numbers of iterations $S(\tilde{w})$	CPU times (sec) $D(\tilde{w})^{-1}L(\tilde{w})$	CPU times (sec) $S(\tilde{w})$
(1e–1,1–1e–1)	1.88	1	1	5.1e–4	5.1e–4
(1e–2,1–1e–2)	2.48	2	1	5.5e–4	4.6e–4
(1e–3,1–1e–3)	2.75	2	1	5.6e–4	4.9e–4
(1e–4,1–1e–4)	2.85	3	2	5.8e–4	5.5e–4
(1e–5,1–1e–5)	2.89	3	2	5.6e–4	5.1e–4
(1e–6,1–1e–6)	2.90	4	2	6.1e–4	5.5e–4
(1e–7,1–1e–7)	2.90	4	2	6.4e–4	5.6e–4

Local uniqueness and minimal positiveness

Theorem 5 (Juang (1995)) $X > 0$ s.t. $(c(1+\alpha)/2)\gamma^T(Xq+e) \leq 1$ and $(c(1-\alpha)/2)\gamma^T(X^Tq+e) \leq 1$ is (globally) unique and minimal positive.

$\Rightarrow$ We check 2 inequalities with the enclosure of X.

Numerical results

$\texttt{M}$: proposed algorithm $(\mathcal{O}(n^2)$ ops. if $\mathcal{O}(1)$ iterations)

$\texttt{V}$: $\texttt{verifynlss}$ for $f(w) = 0$ $(\mathcal{O}(n^3)$ ops.)

maximum radius $= \max_{i,j} X^r_{ij}$, where $\langle \tilde{X}, X^r \rangle \ni X$

$$\text{time ratio} = \frac{\text{CPU time of } \texttt{M} \text{ or } \texttt{V}}{\text{CPU time for computing } \tilde{w}}$$

$\texttt{M}$ and $\texttt{V}$ includes the computation of $\tilde{w} \Rightarrow$ time ratio > 1

If time ratio < 2, verification is faster than the computation of $\tilde{w}$.

Results for various n

$(\alpha, c) = (10^{-1}, 1 - 10^{-1})$

	maximum radii		time ratios	
n	M	V	M	V
256	4.2e–13	8.5e–6	2.54	577
512	4.4e–13	4.6e–5	2.98	2030
1024	1.7e–12	2.5e–4	4.48	3442
2048	1.7e–12	1.4e–3	4.97	8000
4096	6.7e–12	8.2e–3	4.99	17224
8192	1.3e–11	OM	4.97	OM

Results for various (α, c)

$n = 128$

(α, c)	maximum radii M	maximum radii V	time ratios M	time ratios V
(1e–1,1–1e–1)	2.1e–13	1.6e–6	2.53	98.3
(1e–2,1–1e–2)	2.4e–12	6.0e–2	1.95	75.5
(1e–3,1–1e–3)	7.2e–12	NaN	1.24	NaN
(1e–4,1–1e–4)	3.6e–11	NaN	1.16	NaN
(1e–5,1–1e–5)	9.3e–11	NaN	1.08	NaN
(1e–6,1–1e–6)	3.8e–10	NaN	1.01	NaN
(1e–7,1–1e–7)	1.3e–9	NaN	1.00	NaN