Verified numerical computations for blow-up solutions of ODEs

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Blow-up problem of ODEs

Consider the initial value problem defined by the following ordinary differential equations in $\mathbb{R}^m$

$$\frac{dy(t)}{dt} = f\left(y(t)\right), \; y(0) = y_0 \tag{1}$$

where $t \in [0, T)$ with $T \leq \infty$, $f : \mathbb{R}^m \to \mathbb{R}^m$ is a C^1 function (assumed to be a polynomial), and $y_0 \in \mathbb{R}^m$.

Definition Define $t_{\max} > 0$ as

$$t_{\max} := \sup\left\{\bar{t} : \text{a solution } y \in C^1([0, \bar{t})) \text{ of (1) exists}\right\}.$$

We say that the solution y of (1) blows up if $t_{\max} < \infty$.
In such a case, $t_{\max}$ is called the blow-up time of (1).

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Aim

This study gives an answer about

- Solution blows up?
- When do solutions blow up?

by using verified numerical computation, which is based on interval arithmetic.

Keywords:
compactification, desingularization, Lyapunov tracing

Compactifications

"Finite" corresponds to "Infinite"

Definition (Elias-Gingold 2006)

$\mathcal{D} \subset \mathbb{R}^m$: open unit ball.
$T : \mathbb{R}^m \to \mathcal{D}$ is defined by

$$T(y) = x := \frac{y}{\kappa(y)}, \quad \kappa(y) = \kappa(y_1, \cdots, y_m), \tag{2}$$

where $\kappa : \mathbb{R}^m \to \mathbb{R}_{>0} := \{a \in \mathbb{R} : a > 0\}$ is continuous. The map T is called an admissible compactification if

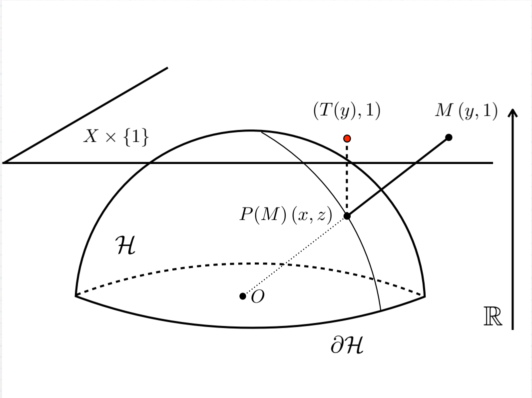
(A0) $\kappa(y) > \|y\|$.

(A1) $\kappa(y) = O(\|y\|)$ as $\|y\| \to \infty$.

(A2) $\nabla\kappa(y) \sim y/\|y\|$ as $\|y\| \to \infty$.

(A3) $\langle y, \nabla\kappa(y)\rangle < \kappa(y)$.

Ex. Poincaré's compactification



$$T(y) := \frac{y}{\kappa(y)},\ \kappa(y) := \sqrt{1 + \|y\|^2}$$

The compactification shows that "Infinity" corresponds to

$$\partial\mathcal{D} := \left\{x \in \mathbb{R}^m \ :\ \|x\|^2 = 1\right\}.$$

Transform (1) via (2).

$$\frac{dx}{dt} = \frac{d}{dt}(y/\kappa(y)) = \kappa^{-1}\frac{dy}{dt} - \kappa^{-2}\left\langle \nabla\kappa(y), \frac{dy}{dt}\right\rangle y$$
$$= \kappa^{-1}[f(y) - \kappa^{-1}\langle\nabla\kappa, f(y)\rangle y]$$
$$= \kappa^{-1}[f(\kappa x) - \langle\nabla\kappa, f(\kappa x)\rangle x],$$

i.e., $\tilde{f}(x,\kappa) := \kappa^{-d} f(\kappa x)$ if $f(y)$ is the d-th degree polynomial

$$\frac{dx}{dt} = [\kappa(T^{-1}(x))]^{d-1}[\tilde{f}(x,\kappa) - \langle\nabla\kappa, \tilde{f}(x,\kappa)\rangle x], \; x(0) = \frac{y_0}{\kappa(y_0)}. \tag{3}$$

There is "singularity at infinity".

On ∂D, $\kappa(T^{-1}(x)) \to \infty$.

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Desingularization

$$\frac{d\tau}{dt} = \kappa(T^{-1}(x))^{d-1}$$

Transform t into τ along a trajectory of $T^{-1}(x)$ by

$$\frac{d\tau}{dt} = \kappa(T^{-1}(x))^{d-1}.$$

We obtain

$$\frac{dx(\tau)}{d\tau} = \tilde{f}\,(x(\tau)) - \langle \nabla\kappa, \tilde{f}\,(x(\tau))\rangle x(\tau),\ \ x(0) = \frac{y_0}{\kappa(y_0)}. \quad (4)$$

Let $g(x) := \tilde{f}\,(x) - \langle \nabla\kappa, \tilde{f}\,(x)\rangle x$. This is the desingularized vector field on $\overline{\mathcal{D}}$.

Note that trajectories of (1) in $\mathbb{R}^m$ and those of (4) in $\mathcal{D}$ have the same topology. We can consider "equilibria" at infinity as follows.

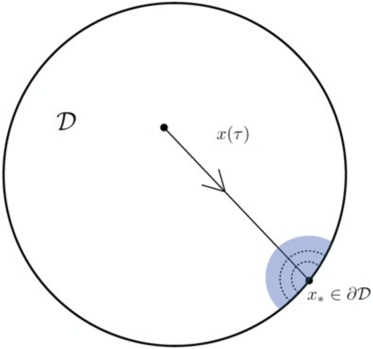
Definition (Elias-Gingold 2006)

- A point sequence $\{y_k\}_{k\geq 1} \subset \mathbb{R}^m$ tends to infinity in the direction y_* $\iff$ $\|y_k\| \to \infty,\ y_k/\|y_k\| \to y_*$ as $k \to \infty$, $\|y_*\| = 1$.
- (1) has a critical point at infinity in the direction x_*, $\|x_*\| = 1$, if x_* is an equilibrium of (4) on $\partial\mathcal{D}$.

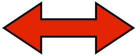
Proposition (Elias-Gingold 2006)

A solution $y(t)$ of (1) has a maximal interval of existence $(a, b) \subset \mathbb{R}$, and $y(t)$ tends to infinity in the direction x_* as $t \to b - 0$ (or as $t \to a + 0$).

$\Rightarrow$ x_* is an equilibrium of (4) on $\partial\mathcal{D}$.



Diverge solutions of $\dot{y} = f(y)$



Global solutions of $\frac{dx}{d\tau} = g(x)$
asymptotic to $x_* \in \partial\mathcal{D}$

Lyapunov tracing

$$t_{\max} = \int_0^\infty \frac{d\tau}{\kappa(T^{-1}(x(\tau)))^{d-1}}$$

Theorem (Matsue-Hiwaki-Yamamoto, arXiv:1604.05953)

- x_*: an equilibrium of (4)
- $N \subset \overline{\mathcal{D}}$: closed, star-shaped, $x_* \in N$
- Dg (Jacobian matrix): continuous in N
- $\exists Y$: m-dim, real symmetric matrix s.t.

$$A(x) := Dg(x)^T Y + Y Dg(x) \tag{5}$$

is strictly negative definite for all $x \in N$.

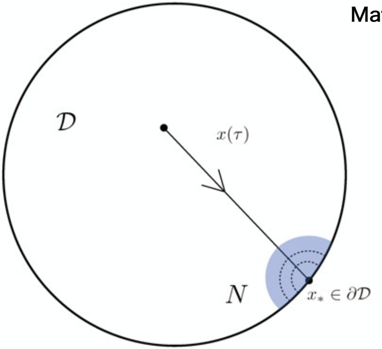
Then,

$$L(x) := (x - x_*)^T Y (x - x_*)$$

is a Lyapunov function defined on N (called a Lyapunov domain).

Lyapunov tracing

Matsue-Hiwaki-Yamamoto, arXiv:1604.05953



$$t_{\max} = \int_0^\infty \frac{d\tau}{\kappa(T^{-1}(x(\tau)))^{d-1}} \quad \Longleftrightarrow \quad t_{\max} = \int_0^{L_0} \cdots \, dL$$

Integral on infinite domain along solution orbits

Integral on finite domain parameterized by level of L

Validation Procedure (Poincaré compactification)

1. Find a critical point at infinity of (1) in the direction x_* and the Lyapunov domain $\tilde{N}$ for (4), i.e., choose $\varepsilon > 0$ so that $N = \left\{x \in \overline{\mathcal{D}} : L(x) \le \varepsilon^2\right\} \subset \tilde{N}$.
2. Solve the ODE (4) and verify $x(\tau_N) \in \operatorname{int} N$. Compute

$$t_N = \int_0^{\tau_N} \frac{d\tau}{\kappa\left(T^{-1}\left(x(\tau)\right)\right)^{d-1}} = \int_0^{\tau_N} \left(1 - \|x(\tau)\|^2\right)^{\frac{d-1}{2}} d\tau$$

   by using verified numerical computations.
3. Compute the estimate

$$\begin{aligned} t_{\max} &\le t_N + \frac{2^{\frac{d-1}{2}}}{c_{\tilde{N}}} \int_0^{L(x(\tau_N))} (c_1 L)^{\frac{d-1}{4}-1} dL \\ &= t_N + \frac{2^{\frac{d-1}{2}} {c_1}^{\frac{d-5}{4}}}{c_{\tilde{N}}} \frac{4}{d-1} L(x(\tau_N))^{\frac{d-1}{4}}, \end{aligned}$$

   where c_1 and $c_{\tilde{N}}$ are computable (bounds of eigenvalues).

Numerical examples

Environments

- OS: Cent OS 6.3
- CPU: Intel(R) Xeon(R) CPU E5-2687W@3.10 GHz
- Library: kv
  C++ Numerical Verification Libraries by Prof. M. Kashiwagi
  `http://verifiedby.me/kv/`
- Compiler: c++
  (gcc version 4.4.7 20120313 (Red Hat 4.4.7-11))
- Options: -O3 -DNDEBUG -DKV_FASTROUND

$dy/dt = y^2$

As a benchmark test, we consider

$$\frac{dy}{dt} = y^2,\ y(0) = a > 0.$$

$t_{\max} = a^{-1}$ (set $a = 1/4$). Desingularized vector field is

$$\frac{dx}{d\tau} = x^2 - x^4,\ x(0) = \frac{a}{\sqrt{1+a^2}}.$$

We set $x_* = 1$ and $N = \{L(x) \leq 1.0 \times 10^{-20}\} \cap \overline{\mathcal{D}}$. When $\tau_N = 15$, we enclosed $x(\tau_N) \in [0.999999999941121, 0.999999999941125] \subset N$

$$t_{\max} \in [3.9999713430726937, 4.0000178056561886].$$

This actually includes the exact blow-up time $t_{\max} = 4$. The execution time was about 0.104 seconds.

Two-dimensional system

$$\begin{cases} \dfrac{dy_1}{dt} = y_1^2 + y_2^2 - 1, \\ \dfrac{dy_2}{dt} = 5(y_1 y_2 - 1), \end{cases}$$

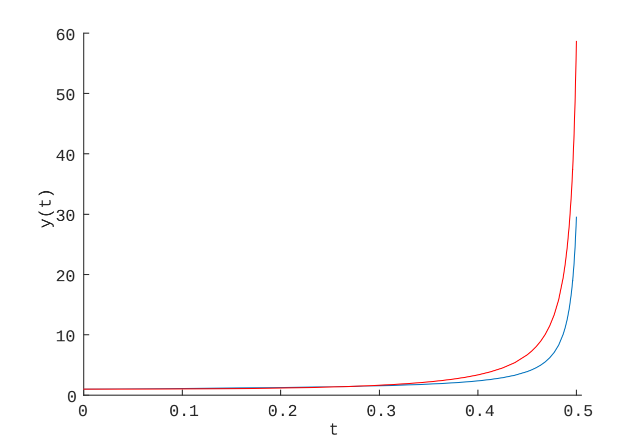
$y = (y_1, y_2)^T$, $y(0) = (1, 1)^T$. Desingularized vector field is

$$\begin{cases} \dfrac{dx_1}{d\tau} = (1 - x_1^2)(2(x_1^2 + x_2^2) - 1) - 5x_1 x_2 (x_1^2 + x_1 x_2 + x_2^2 - 1), \\ \dfrac{dx_2}{d\tau} = 5(1 - x_2^2)(x_1^2 + x_1 x_2 + x_2^2 - 1) - x_1 x_2 (2(x_1^2 + x_2^2) - 1). \end{cases}$$

$x_* = (1/\sqrt{5}, 2/\sqrt{5})$, $N = \{L(x) \leq 1.0 \times 10^{-20}\} \cap \overline{\mathcal{D}}$. $\tau_N = 7$:

$$x(\tau_N) \in \begin{pmatrix} [0.447213595573401, 0.447213595573404] \\ [0.894427190963148, 0.894427190963151] \end{pmatrix} \subset N$$

$$t_{\max} \in [0.50680733588232473, 0.50682093902984382].$$



Note that all the components of the solution y (y_1: blue, y_2: red) blow up at the same time because

$$y_i = \frac{x_i}{(1 - \|x\|^2)^{1/2}} \quad \text{for all } i.$$

The execution time was about 1.217 seconds.

Discretized nonlinear heat equation

Finite difference semi-discretization of

$$\begin{cases} \partial_t u = \Delta u + u^3, & t > 0,\ x \in (0,1), \\ u = 0 & \text{at } x = 0, 1 \end{cases}$$

Grid number: $n+1$, initial data $y_k(0) = 10$ $(k = 1, 2, \ldots, n-1)$

n	τ_N	$t_{\max}$	exec. time
3	30	$0.0050340400^{784869202}_{162383761}$	44.417s
5	30	$0.00500977^{25547564119}_{0457049421}$	3m15.793s
7	30	$0.005003^{9439361921953}_{7433760869625}$	7m11.506s
9	35	$0.005001^{9593060980734}_{7211768978893}$	20m56.604s
11	40	$0.00500^{12269722264098}_{08814990989457}$	72m10.282s
13	50	$0.005000^{6415371753669}_{4463436608779}$	202m0.581s
15	50	$0.00500^{12389304747536}_{0091354528135}$	387m26.199s

Conclusion

Computable "infinity"

- "Infinity" in phase space is transformed into boundary of compact manifold (compactification)
- "Finite-time" singularity is resolved by desingularization ("infinite-time" behavior)
- Re-parameterization of the integral on infinite domain (Lyapunov tracing)

We can treat blow-up solutions by using verified numerical computations.

For more detail, see arXiv : 1606.03039

Thank you for kind attention!