On verification methods for parabolic partial differential equations using the evolution operator

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Joint with

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Semilinear heat eq.

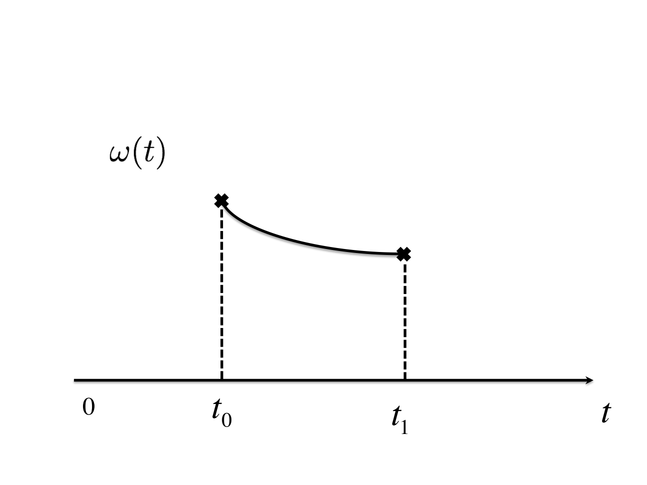
$J := (t_0, t_1]$ $(0 \le t_0 < t_1 < \infty)$,

$\Omega = (0,1)^d \subset \mathbb{R}^d$ $(d = 1,2,3)$,

$$(P) \begin{cases} \partial_t u - \Delta u = u^p & \text{in } J \times \Omega, \\ u(t,x) = 0, & t \in J,\ x \in \partial\Omega, \\ u(t_0, x) = u_0(x), & x \in \Omega. \end{cases}$$

- $\partial_t u = \frac{\partial u}{\partial t}$, $\Delta = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_d^2}$, $D(\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$;
- $u_0 \in L^2(\Omega)$ is an initial function;
- $1 < p < 1 + \frac{4}{d}$;
- $\tau := t_1 - t_0$.

The 1st Step: Local inclusion in a Banach space X



Construct $\omega(t)$ based on numerical solutions (any).

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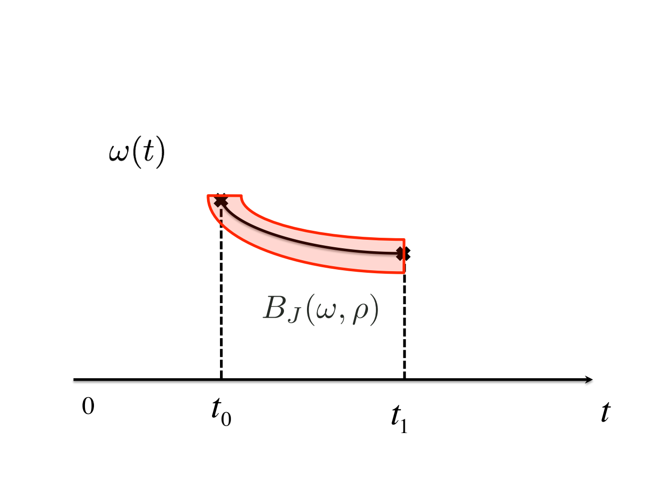

Fixed-point th. derives a local inclution of the exact solution.

Local inclusion

X, Y : Banach spaces; ω: approximate solution (numerically);
Assuming that the initial function satisfies
$\|u_0 - \omega(t_0)\|_Y \le \varepsilon_0$, we rigorously enclose the solution of (P) in X. Namely, we compute a radius $\rho > 0$ of the ball:

$$B_J(\omega, \rho) := \{y \in X : \|y - \omega\|_X \le \rho\}.$$

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$$X = C(J; D(\Delta_\mu^\alpha)),\ Y = L^2(\Omega),\ u_0 \in L^2(\Omega),$$

where $D(\Delta_\mu^\alpha)$ is a fractional power of the shifted positive operator[1] with $\mu > 0$.

The evolution operator is a solution operator of a homogenious Cauchy problem:

$$\begin{cases} \partial_t u + A(t)u = 0 \\ u(s) = \phi, \end{cases}$$

where $A(t) := -\Delta + (\sigma - p\omega(t)^{p-1})$ and $\sigma > 0$ satisfies

$$\sigma - p\omega(t,x)^{p-1} \geq \mu, \quad t \in J, \text{ a.e. } x \in \Omega.$$

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Analytic semigroup (2014)

For $z = u - \omega$, we define an fixed-point form:

$$(S(z))(t) := e^{-tA} z_0 + \int_{t_0}^{t} e^{-(t-s)A} \tilde{g}(z(s)) ds \quad (t_0 \le s \le t \le t_1),$$

where $z_0 := z(t_0)$, $\tilde{g}(z) = (z+\omega)^p - \omega^p - \partial_t \omega + \Delta\omega + \omega^p$.

Evolution operator

For a fixed $\sigma > 0$, let $z = e^{\sigma(t-t_0)} v$. We define

$$(T(v))(t) := U(t,t_0) v_0 + \int_{t_0}^{t} U(t,s) g(v(s)) ds \quad (t_0 \le s \le t \le t_1),$$

where $v_0 := v(t_0)$, $g(v) =$

$$e^{-\sigma(t-t_0)} \left\{ \left(e^{\sigma(t-t_0)} v + \omega \right)^p - \omega^p - p\omega(t)^{p-1} e^{\sigma(t-t_0)} v - \partial_t \omega + \Delta\omega + \omega^p \right\}.$$

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Let X_α be a weighted subspace of $C(J; D(\Delta_\mu^\alpha))$ defined by

$$X_\alpha := \left\{ u : \sup_{t \in J} (t - t_0)^\alpha \|\Delta_\mu^\alpha u\|_{L^2} < +\infty \right\}$$

with the norm $\|\cdot\|_{X_\alpha} := \sup_{t\in J}(t-t_0)^\alpha \|\Delta_\mu^\alpha \cdot\|_{L^2}$. This X_α becomes a Banach space[2] with the norm $\|\cdot\|_{X_\alpha}$.

A sufficient condition is given for guaranteeing existence and local uniqueness of an exact solution in

$$B_J(\omega, \rho) := \left\{ u : \sup_{t \in J} (t - t_0)^\alpha e^{-\sigma(t-t_0)} \|\Delta_\mu^\alpha (u - \omega)\|_{L^2} \le \rho \right\}$$

for a fixed $\alpha \in (0, 1)$.

[2]Since the embedding $D(\Delta_\mu^\alpha) \hookrightarrow L^2(\Omega)$ exists, the norm $\|\cdot\|_{X_\alpha}$ is equivalent to the graph norm of $C(J; D(\Delta_\mu^\alpha))$.

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Theorem (Local inclusion)

Let α satisfy $\alpha \in (\frac{d(p-1)}{4p}, \frac{1}{p})$. Assuming that an approximate solution ω satisfies $\|u_0 - \omega(t_0)\|_{L^2} \le \varepsilon_0$ and

$$\|\partial_t \omega - \Delta\omega - f(\omega)\|_{C(J;L^2(\Omega))} \le \delta.$$

If

$$W(\tau)\left(\varepsilon_0 + L_\omega(\rho)\rho^2 + \frac{\delta\tau}{1-\alpha}\right) < \rho$$

holds for some $\rho > 0$, then the solution $u(t) := u(t,\cdot)$ $(t \in J)$ of (P) uniquely exists in the ball $B_J(\omega,\rho)$.

Here, $W(\tau)$ and $L_\omega(\rho)$ are given by

$$W(\tau) := \left(\frac{\alpha}{e}\right)^\alpha \left\{1 + \frac{C_\omega \tau^2}{(1-\alpha)(2-\alpha)}\right\},$$

$$
\begin{aligned}
L_\omega(\rho) :=& p(p-1)C_{2p,\alpha}^2 e^{\sigma\tau} \cdot \\
& \left(\tau^\alpha \|\omega\|_{C(J;L^{2p}(\Omega))} + C_{2p,\alpha} e^{\sigma\tau}\rho\right)^{p-2} \cdot \\
& \tau^{1-p\alpha} B(1-\alpha, 1-p\alpha),
\end{aligned}
$$

respectively, if $A(t)$ satisfies

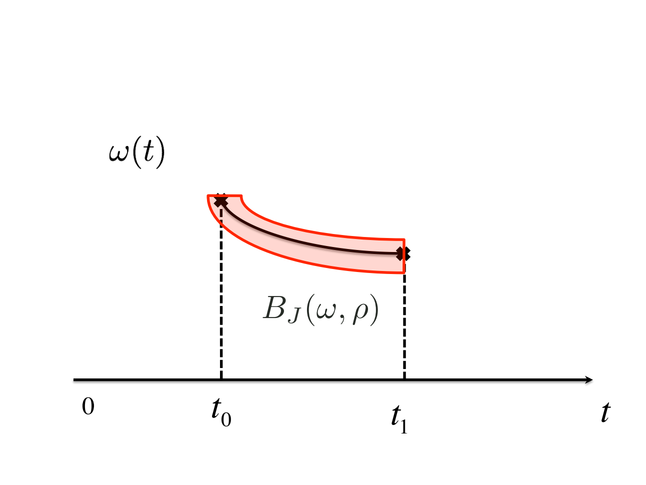
$$
\|A(t) - A(s)\| \leq C_\omega(t-s),\ s \leq t.
$$

Furthermore, e denotes the Euler number, $C_{2p,\alpha}$ is an embedding constant satisfying

$$
\|\phi\|_{L^{2p}} \leq C_{2p,\alpha} \|\Delta_\mu^\alpha \phi\|_{L^2},\ \forall \phi \in D\left(\Delta_\mu^\alpha\right),
$$

and $B(x, y)$ is the Beta function.

The 2nd Step: Error estimate at $t = t_1$



After getting the local inclusion we estimate

$$\|u(t_1) - \omega(t_1)\|_{L^2} \leq \varepsilon_1.$$

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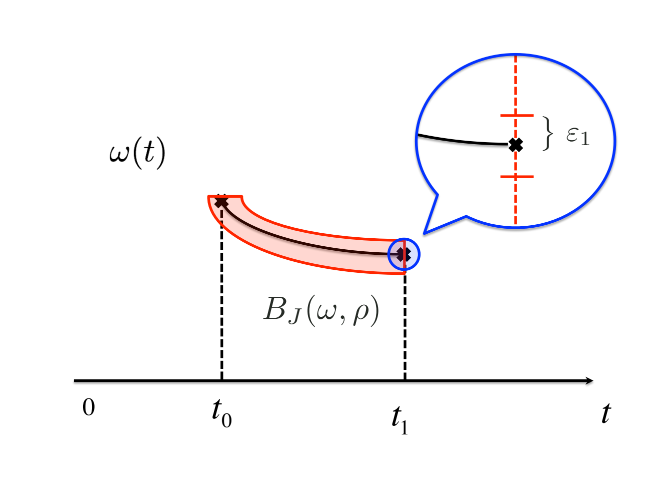

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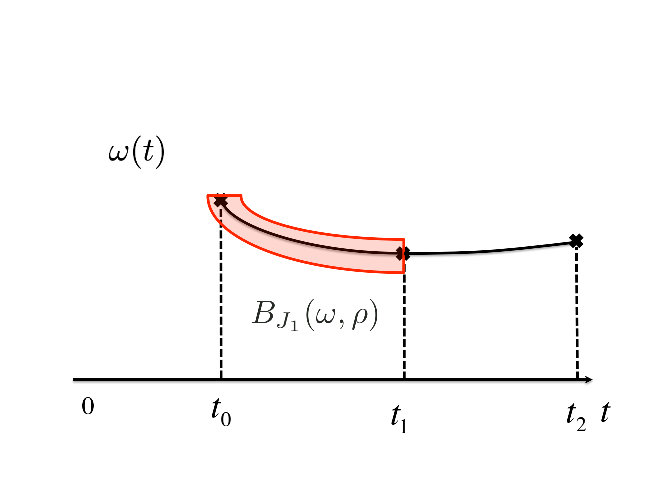

Then extend the approximate solution $\omega(t)$ in $J_2 = (t_1, t_2]$.

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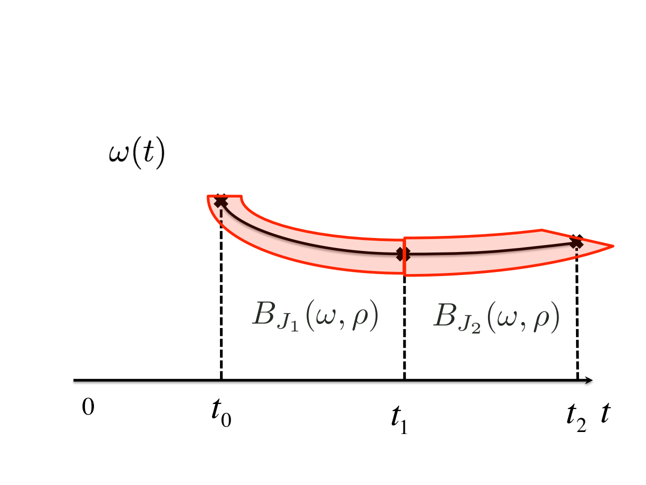

In J_2 the local inclusion theorem is checked again numerically.
Then extend the inclusion of the exact solution in $J_1 \cup J_2$.

Concatenation scheme

Let n be a natural number and $0 = t_0 < t_1 < \cdots < t_n < \infty$. We denote $J_i = (t_{i-1}, t_i]$ and $\tau_i = t_i - t_{i-1}$ $(i = 1, 2, ..., n)$. σ_i and ρ_i are determined by the local inclusion theorem in each J_i.

Assuming that the exact solution is rigorously enclosed until J_n, i.e., it holds in each J_i $(i = 1, 2, ..., n)$

$$u \in B_{J_i}(\omega, \rho_i) = \left\{ u : \sup_{t \in J_i} (t - t_{i-1})^{\alpha} e^{-\sigma_i (t - t_{i-1})} \|\Delta_\mu^\alpha (u - \omega)\|_{L^2} \le \rho_i \right\},$$

we aim to bound the error estimate at $t = t_n$ such that

$$\|u(t_n) - \omega(t_n)\|_{L^2} \le \varepsilon_n.$$

Task is avoiding the propagation[3] of the error in order to enclose the solution for a long time.

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Let $z = u - \omega$ for $t \in J_i$. The z is the mild solution of the following parabolic equation:

$$\begin{cases} \partial_t z + B(t)z = h_i(z), & t \in J_i, x \in \Omega, \\ z(t,x) = 0, & t \in J_i, x \in \partial\Omega, \\ z(t_{i-1}, x) = u(t_{i-1}) - \omega(t_{i-1}), & x \in \Omega, \end{cases}$$

where

$$\begin{aligned} B(t) &= -\Delta - p\omega(t)^{p-1} \\ h_i(z) &= (z+\omega)^p - \omega^p - p\omega^{p-1}z + \omega^p - \partial_t\omega + \Delta\omega. \end{aligned}$$

Since $A(t)$ generates the evolution operator and $A(t) = B(t) + \sigma_i$, the real perturbed operator $B(t)$ also generates the evolution operator $\{U_B(t,s)\}_{t_{i-1} \le s < t \le t_i}$ for J_i. Then, it holds for $t \in J_i$

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Lemma

For $t, s \in J_i$, it follows

$$\|U_B(t,s)\phi\|_{L^2} \le e^{-(\lambda_A - \sigma_i)(t-s)}\|\phi\|_{L^2},\ \forall \phi \in L^2(\Omega),$$

where λ_A denotes the lower bound of the minimal eigenvalue of $A(t)$ for $t \in J_i$.

Then we have the following estimates recursively:

$$\|z(t_0)\|_{L^2} = \|u_0 - \omega(t_0)\|_{L^2} \le \varepsilon_0,$$

$$\begin{aligned}\|z(t_1)\|_{L^2} &\le \|U_B(t_1,t_0)z(t_0)\|_{L^2} + \int_{J_1} \|U_B(t_1,s)h_1(z(s))\|_{L^2} ds \\ &\le e^{-(\lambda_A-\sigma_1)(t_1-t_0)}\|z(t_0)\|_{L^2} + \int_{J_1} \|U_B(t_1,s)h_1(z(s))\|_{L^2} ds \\ &\le e^{-(\lambda_A-\sigma_1)T_1}\varepsilon_0 + \nu_1.\end{aligned}$$

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We repeat this estimate.

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\end{aligned}
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\|z(t_3)\|_{L^2} &\leq \|U_B(t_3,t_2)z(t_2)\|_{L^2} + \int_{J_3} \|U_B(t_3,s)h_3(z(s))\|_{L^2}\,ds \\
&\leq e^{-(\lambda_A-\sigma_3)(t_3-t_2)}\|z(t_2)\|_{L^2} + \int_{J_3} \|U_B(t_3,s)h_3(z(s))\|_{L^2}\,ds \\
&\leq e^{-(\lambda_A-\sigma_3)\tau_3}\left(\left(e^{-(\lambda_A-\sigma_2)\tau_2}e^{-(\lambda_A-\sigma_1)\tau_1}\varepsilon_0\right) + e^{-(\lambda_A-\sigma_2)\tau_2}\nu_1 + \nu_2\right) + \nu_3 \\
&= \left(e^{-(\lambda_A-\sigma_3)\tau_3}e^{-(\lambda_A-\sigma_2)\tau_2}e^{-(\lambda_A-\sigma_1)\tau_1}\right)\varepsilon_0 \\
&\quad + \left(e^{-(\lambda_A-\sigma_3)\tau_3}e^{-(\lambda_A-\sigma_2)\tau_2}\right)\nu_1 + e^{-(\lambda_A-\sigma_3)\tau_3}\nu_2 + \nu_3
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
\|z(t_n)\|_{L^2} \le & \left(e^{-(\lambda_A-\sigma_n)\tau_n} e^{-(\lambda_A-\sigma_{n-1})\tau_{n-1}} \cdots e^{-(\lambda_A-\sigma_1)\tau_1}\right) \varepsilon_0 \\
& + \left(e^{-(\lambda_A-\sigma_n)\tau_n} e^{-(\lambda_A-\sigma_{n-1})\tau_{n-1}} \cdots e^{-(\lambda_A-\sigma_2)\tau_2}\right) \nu_1 \\
& + \left(e^{-(\lambda_A-\sigma_n)\tau_n} e^{-(\lambda_A-\sigma_{n-1})\tau_{n-1}} \cdots e^{-(\lambda_A-\sigma_3)\tau_3}\right) \nu_2 \\
& + \cdots + e^{-(\lambda_A-\sigma_n)\tau_n} \nu_{n-1} + \nu_n.
\end{aligned}
$$

By estimating the inside of the bracket first, we can shrink the error propagation.

These estimates are imitation of avoiding the wrapping effect in verification methods for ODEs.

Finally, we have

$$\begin{aligned}\|z(t_n)\|_{L^2} \le & \left(e^{-(\lambda_A-\sigma_n)\tau_n} e^{-(\lambda_A-\sigma_{n-1})\tau_{n-1}} \cdots e^{-(\lambda_A-\sigma_1)\tau_1}\right) \varepsilon_0 \\ & + \left(e^{-(\lambda_A-\sigma_n)\tau_n} e^{-(\lambda_A-\sigma_{n-1})\tau_{n-1}} \cdots e^{-(\lambda_A-\sigma_2)\tau_2}\right) \nu_1 \\ & + \left(e^{-(\lambda_A-\sigma_n)\tau_n} e^{-(\lambda_A-\sigma_{n-1})\tau_{n-1}} \cdots e^{-(\lambda_A-\sigma_3)\tau_3}\right) \nu_2 \\ & + \cdots + e^{-(\lambda_A-\sigma_n)\tau_n} \nu_{n-1} + \nu_n.\end{aligned}$$

By estimating the inside of the bracket first, we can shrink the error propagation.

These estimates are imitation of avoiding the wrapping effect in verification methods for ODEs.

Fujita-type equation

$\Omega := (0,1)^2$: Unit square domain in $\mathbb{R}^2$, $T > 0$

$$(F)\ \begin{cases} \partial_t u - \Delta u = u^2 & \text{in } (0,T)\times\Omega, \\ u(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $u_0(x) = \gamma \sin(\pi x_1)\sin(\pi x_2)$.

OS: Cent OS 6.3

CPU: Intel(R) Xeon(R) CPU E5-2687W @ 3.10 GHz

MATLAB 2012b with INTLAB ver.7.1

$V_N := \left\{ \sum_{k,l=1}^{N} a_{k,l} \sin(k\pi x_1)\sin(l\pi x_2) \ : \ a_{k,l} \in \mathbb{R} \right\}$;

Approximate solution

- γ is a parameter.
- For a sufficiently large γ, the solution will blows up in finite time.
- The large solution is difficult for verification method.

- Employ Fourier-Galerkin method in space
- Crank-Nicolson scheme in time
- The ω is defined by

$$\omega(t,x) = \sum_{|m|\leq N^2} u_m(t)\psi_m(x),$$

where $m = (m_1, m_2)$ is a multi-index and $\psi_m(x) = \sin(m_1\pi x_1)\sin(m_2\pi x_2)$.

Prev. (2014) v.s. New

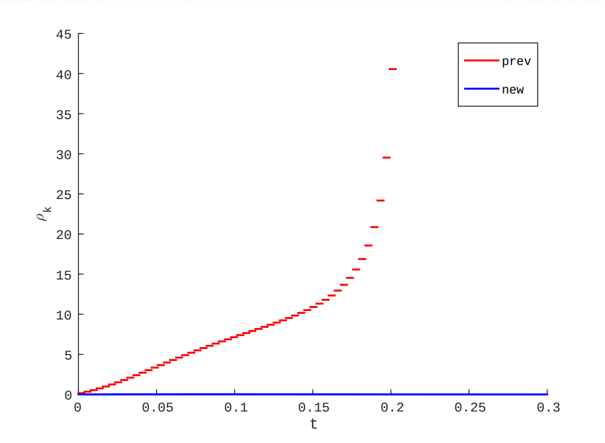

Fig. $\gamma = 6.8$, $\alpha = 1/2$ (prev.), $3/8$ (new), $\mu = 70$, $N = 5$.

Result of inclusion using the evolution operator is much tighter than that of the previous one (analytical semigroup).

Shrink technique

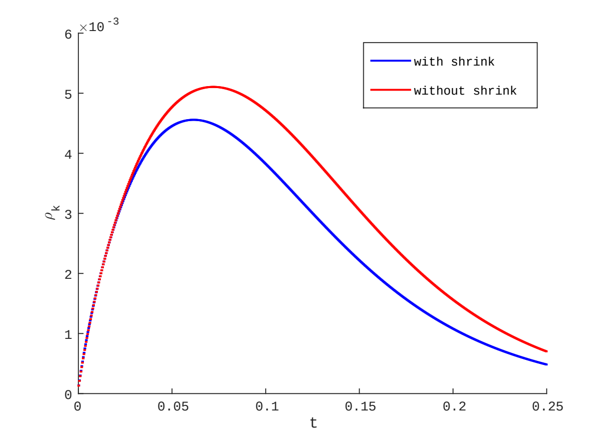

Fig. $\gamma = 7$, $\alpha = 3/8$, $\mu = 70$, $N = 5$.

Result using the shrink technique gives tighter enclosure.

Shrink technique

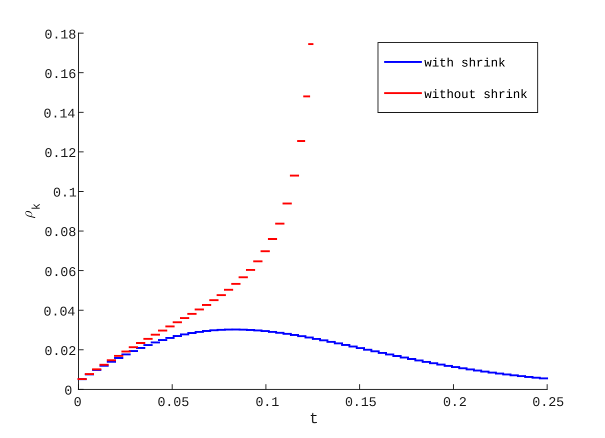

Fig. $\gamma = 7$, $\alpha = 3/8$, $\mu = 70$, $N = 5$.

Result using the shrink technique gives tighter enclosure.

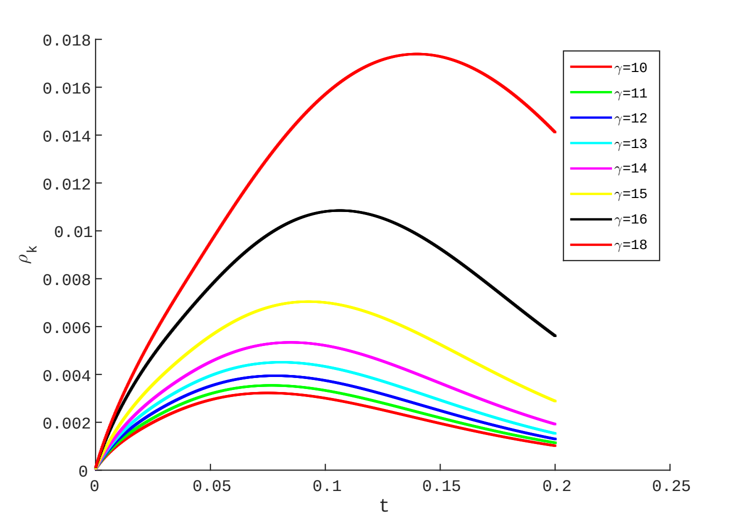

Fig. Result varying γ

$(\alpha = 3/8,\ \mu = 600\,(\gamma < 15),\ 550\,(\gamma = 15),\ 350\,(\gamma > 15),\ N = 11)$.

Thank you for kind attention!