Error Analysis of Lagrange Interpolation on Tetrahedrons

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The kth-order Lagrange interpolation on tetrahedrons

k : a positive interger,

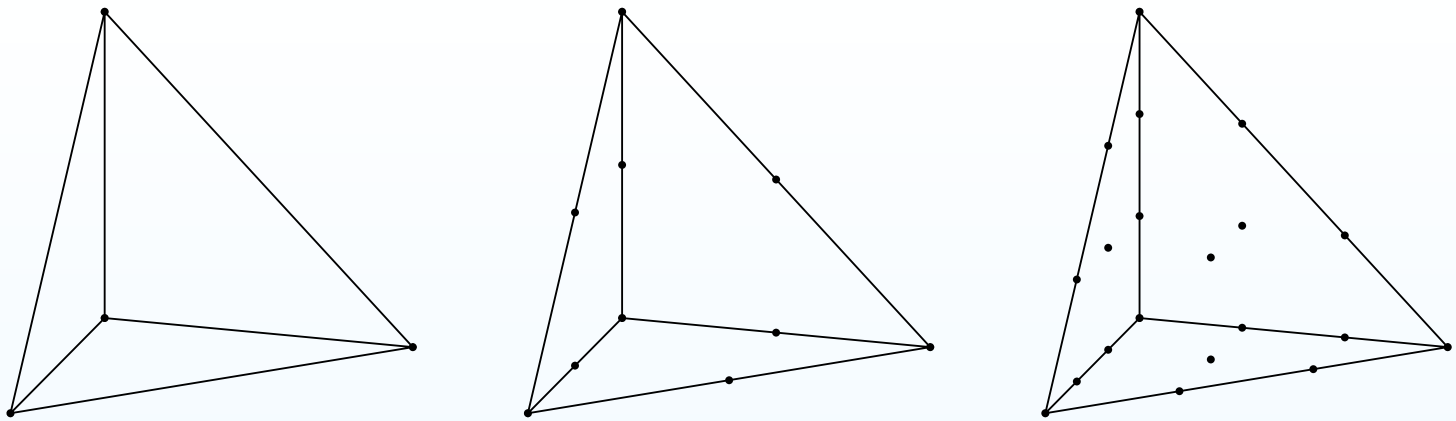
$\mathcal{P}_k$: the set of polynomials whose order are at most k,

$K \subset \mathbb{R}^3$: any tetrahedron in $\mathbb{R}^3$,

$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$: the barycentric coordinate on K,

a_i : integers,

$$\Sigma^k(K) := \left\{ \left(\frac{a_1}{k}, \cdots, \frac{a_4}{k} \right) \in K \;\middle|\; 0 \le a_i \le k,\ \sum_{i=1}^{4} a_i = k \right\}.$$



K and $\Sigma^k(K)$, $k=1$, $k=2$, $k=3$.

For $v \in C^0(K)$, define $\mathcal{I}_K^k v \in \mathcal{P}_k$ by

$$(\mathcal{I}_K^k v)(\mathbf{x}) = v(\mathbf{x}), \qquad \forall \mathbf{x} \in \Sigma^k(K).$$

An important thing is to obtain an error estimation such as

$$|v - \mathcal{I}_K^k v|_{m,p,K} \le C h_K^{k+1-m} |v|_{k+1,p,K}.$$

The piecewise $\mathcal{P}_k$ finite element method

$\Omega \subset \mathbb{R}^d, d = 1, 2, 3$: a bounded polygonal domain

τ : a proper (face-to-face) triangulation of Ω

$$S_h := \left\{ v_h \in C^0(\overline{\Omega}) \cap H_0^1(\Omega) \;\middle|\; v|_K \in \mathcal{P}_k, \forall K \in \tau \right\}$$

Set of piecewise linear functions on τ

Model problem Find $u \in H_0^1(\Omega)$ such that

$-\Delta u = f$ for a given $f \in L^2(\Omega)$.

Weak form Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \mathrm{d}x = \int_\Omega f v \mathrm{d}x \quad \text{for } \forall v \in H_0^1(\Omega).$$

P_k FEM Find $u_h \in S_h$ such that

$$\int_\Omega \nabla u_h \cdot \nabla v_h \mathrm{d}x = \int_\Omega f v_h \mathrm{d}x \quad \text{for } \forall v_h \in S_h.$$

Let $u \in H_0^1(\Omega) \cap H^2(\Omega)$ and $u_h \in S_h$ be the exact and finite element solutions, respectively. Then, by Céa's Lemma, we have

$$\begin{aligned} \|u - u_h\|_{1,2,\Omega} &\le C \inf_{v_h \in S_h} \|u - v_h\|_{1,2,\Omega} \\ &\le C \|u - \mathcal{I}_\tau^k u\|_{1,2,\Omega} \\ &= C \left(\sum_{K \in \tau} \|u - \mathcal{I}_K^k u\|_{1,2,K}^2 \right)^{1/2}, \end{aligned}$$

where C is a positive constant.

Therefore, estimating $\|u - \mathcal{I}_K^k u\|_{1,2,K}$ is very important for the error analysis of the finite element methods.

The reference tetrahedrons

Let $\widehat{K}$ and $\widetilde{K}$ be the tetrahedrons that have the following vertices:

$$\widehat{K} \text{ has the vertices } (0,0,0)^\top,\ (1,0,0)^\top,\ (0,1,0)^\top,\ (0,0,1)^\top,$$
$$\widetilde{K} \text{ has the vertices } (0,0,0)^\top,\ (1,0,0)^\top,\ (1,1,0)^\top,\ (0,0,1)^\top.$$

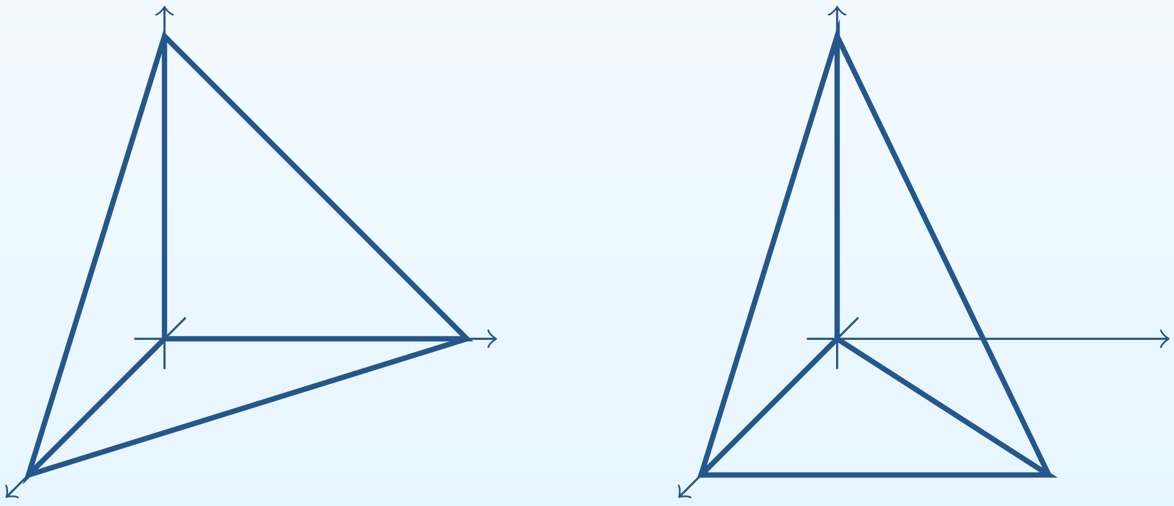


Figure 1: $\widehat{K}$ and $\widetilde{K}$.

We denote the reference tetrahedron by $\mathbf{K}$, that is, $\mathbf{K}$ is either $\widehat{K}$ or $\widetilde{K}$.

The Basic Idea

An arbitrary triangle $K \subset \mathbb{R}^3$ can be obtained by a affine transformation $\varphi_K(\mathbf{x}) := A\mathbf{x} + \mathbf{b}$ as $K = \varphi_K(\mathbf{K})$.

The important factors are $\|A\|$ and $\|A^{-1}\|$. It seems that

> if K becomes very "flat", then the estimation would become very "poor."

It seems that we need a geometric condition on K to obtain an error estimation.

The Standard Error Estimation

Let $h_K := \mathrm{diam} K$ and ρ_K be the diameter of the inscribed sphere of K.

Theorem 1 *Let $\gamma > 0$ be a constant. If $h_K/\rho_K \le \gamma$, there exists a constant $C = C(\gamma)$ independent of h_K such that*

$$|v - \mathcal{I}_K^k v|_{m,p,K} \le C h_K^{k+1-m} |v|_{k+1,p,K}, \qquad \forall v \in W^{k+1,p}(K).$$

Ciarlet, *The Finite Element Methods for Elliptic Problems,*
North Holland, 1978, reprint by SIAM 2008.
Brenner-Scott, *The Mathematical Theory of Finite Element Methods,* 3rd ed.
Springer, 2008.

If a triangulation τ satisfies $\max_{K \in \tau} h_K/\rho_K \le C$, τ is called regular. The value $\max_K h_K/\rho_K$ is called the **chunkiness parameter**.

Křížek's maximum angle condition

Theorem 2 *Let* θ_2 $(\pi/3 \le \theta_2 < \pi)$ *be a constant. Let* γ_K *be the maximum angle of faces of a tetrahedron* K *and* φ_K *be the maximum angle between faces of* K*. If* $\gamma_K \le \theta_2$*,* $\varphi_K \le \theta_2$*, and* $h_K \le 1$*, then there exists a constant* $C = C(\theta_2)$ *that is independent of* h_K *such that*

$$\|v - \mathcal{I}_K^1 v\|_{1,p,K} \le C h_K |v|_{2,p,K}, \qquad \forall v \in W^{2,p}(K), \quad 2 < p \le \infty.$$

M. Křížek, On the maximum angle condition for linear tetrahedral elements, SIAM J. Numer. Anal., **29** (1992), 513–520.

R.G. Durán, Error estimates for 3-d narrow finite elements, Math. Comp., **68** (1999), 187–199.

The Circumradius estimation

Theorem 3 (Kobayashi-Tsuchiya) *Let $K \subset \mathbb{R}^2$ be an arbitrary triangle. Let R_K be the circumradius of K and $h_K := \mathrm{diam} K$. For any positive integer k and p, $1 \le p \le \infty$, there exists a constant $C_{k,p}$ independent of K such that, for $m = 0, 1, \cdots, k$ and $\forall v \in W^{k+1,p}(K)$,*

$$
\begin{aligned}
|v - \mathcal{I}^k v|_{m,p,K} &\le C_{k,p} R_K^m h_K^{k+1-2m} |v|_{k+1,p,K} \\
&= C_{k,p} \left(\frac{R_K}{h_K} \right)^m h_K^{k+1-m} |v|_{k+1,p,K}.
\end{aligned}
$$

Note that *no geometric condition is imposed on K*.

Kobayashi, Tsuchiya, *A priori* error estimates for Lagrange interpolations on triangles, Applications of Mathematics, **60** (2015) 485–499.
Kobayashi, Tsuchiya, Extending Babuška-Aziz's theorem to higher-order Lagrange interpolation, Applications of Math., **61** (2016) 121–133.

The error estimation of Lagrange interpolation

Let $\mathcal{T}_p^k(K)$ and $B_p^{m,k}(K)$ be defined by

$$\mathcal{T}_p^k(K) := \left\{ v \in W^{k+1,p}(K) \;\middle|\; v(\mathbf{x}) = 0,\ \forall \mathbf{x} \in \Sigma^k(K) \right\},$$

$$B_p^{m,k}(K) := \sup_{v \in \mathcal{T}_p^k(K)} \frac{|v|_{m,p,K}}{|v|_{k+1,p,K}}.$$

From the definitions, we have $v - \mathcal{I}_K^k v \in \mathcal{T}_p^k(K)$ for any $v \in W^{k+1,p}(K)$ and

$$|v - \mathcal{I}_K^k v|_{m,p,K} \le B_p^{m,k}(K)|v|_{k+1,p,K}.$$

Note that

$$B_p^{m,k}(K) = \inf \left\{ C;\, |v - \mathcal{I}_K^k v|_{m,p,K} \le C|v|_{k+1,p,K},\ \forall v \in W^{k+1,p}(K) \right\}.$$

That is, $B_p^{m,k}(K)$ is the *best* constant for the error estimation

$$|v - \mathcal{I}_K^k v|_{m,p,K} \le C|v|_{k+1,p,K}.$$

The projected circumradius of tetrahedrons

Let K be an arbitrary tetrahedron and B be any facet of K. We regard B as the base of K.

Consider any plane H perpendicular to B and the orthogonal projection δ_H on H. The image $\delta_H(K)$ is a triangle on P. Let

$$R_P := \max_H \{\text{circumradius of } \delta_H(K)\}.$$

The **projected circumradius** R_K of a tetrahedron K is defined by

$$R_K := \min_B \frac{R_B R_P}{h_B}, \tag{1}$$

where the minimum is taken over all the facets of K.

The main theorem

Theorem 4 *Let K be an arbitrary tetrahedron. Let $h_K := \mathrm{diam} K$ and R_K be the projected circumradius of K. Assume that k, m are integers with $k \geq 1$, $0 \leq m \leq k$, and p is taken as*

$$k - m = 0 \Longrightarrow 2 < p \leq \infty,$$
$$k = 1,\ m = 0 \Longrightarrow \frac{3}{2} < p \leq \infty,$$
$$k \geq 2,\ k - m \geq 1 \Longrightarrow 1 \leq p \leq \infty.$$

Then, for arbitrary $v \in W^{k+1,p}(K)$, there exists a constant $C = C(k,m,p)$ independent of K such that

$$\begin{aligned} |v - \mathcal{I}_K^k v|_{m,p,K} &\leq C R_K^m h_K^{k+1-2m} |v|_{k+1,p,K} \\ &= C \left(\frac{R_K}{h_K} \right)^m h_K^{k+1-m} |v|_{k+1,p,K}. \end{aligned}$$

Note that *no geometric condition is imposed on K*.

The reference tetrahedrons

Let $\widehat{K}$ and $\widetilde{K}$ be the tetrahedrons that have the following vertices:

$$\widehat{K} \text{ has the vertices } (0,0,0)^\top,\ (1,0,0)^\top,\ (0,1,0)^\top,\ (0,0,1)^\top,$$
$$\widetilde{K} \text{ has the vertices } (0,0,0)^\top,\ (1,0,0)^\top,\ (1,1,0)^\top,\ (0,0,1)^\top.$$

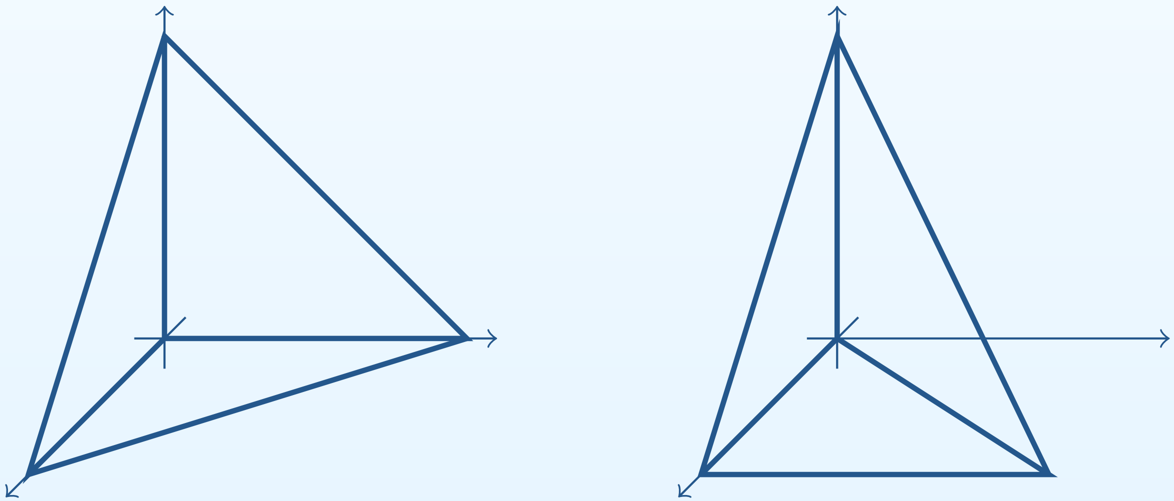


Figure 2: $\widehat{K}$ and $\widetilde{K}$.

We denote the reference tetrahedron by $\mathbf{K}$, that is, $\mathbf{K}$ is either $\widehat{K}$ or $\widetilde{K}$.

The Squeezing Theorem

We now generalize the squeezing map. Let α, β, and $\gamma \in \mathbb{R}$ be positive. We then define the *squeezing map* $sq_2^{\alpha\beta\gamma} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$sq_2^{\alpha\beta\gamma}(x, y, z) := (\alpha x, \beta y, \gamma z)^\top, \quad (x, y, z)^\top \in \mathbb{R}^3.$$

Theorem 5 *Let* $K_{\alpha\beta\gamma} := sq_2^{\alpha\beta\gamma}(\mathbf{K})$*. Assume that* $k \geq 1$*,* $0 \leq m \leq k$*, and* p *is taken as*

$$k - m = 0 \Longrightarrow 2 < p \leq \infty,$$
$$k = 1,\ m = 0 \Longrightarrow \frac{3}{2} < p \leq \infty,$$
$$k \geq 2,\ k - m \geq 1 \Longrightarrow 1 \leq p \leq \infty.$$

We then have

$$B_p^{m,k}(K_{\alpha\beta\gamma}) := \sup_{v \in \mathcal{T}_p^k(K_{\alpha\beta\gamma})} \frac{|v|_{m,p,K_{\alpha\beta\gamma}}}{|v|_{k+1,p,K_{\alpha\beta\gamma}}} \leq (\max\{\alpha, \beta, \gamma\})^{k+1-m} C_{k,m,p}.$$

The Standard Position for Tetrahedrons

Let K be a tetrahedron with vertices $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$.
Let B be the facet of K with vertices $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ $\Longleftarrow$ the base of K.
Let α, β, $0 < \beta \le \alpha$, be the longest and shortest lengths of the edges of B.
We assume that $\mathbf{x}_1\mathbf{x}_2$ is the longest edge of B; $|\mathbf{x}_1 - \mathbf{x}_2| = \alpha$.

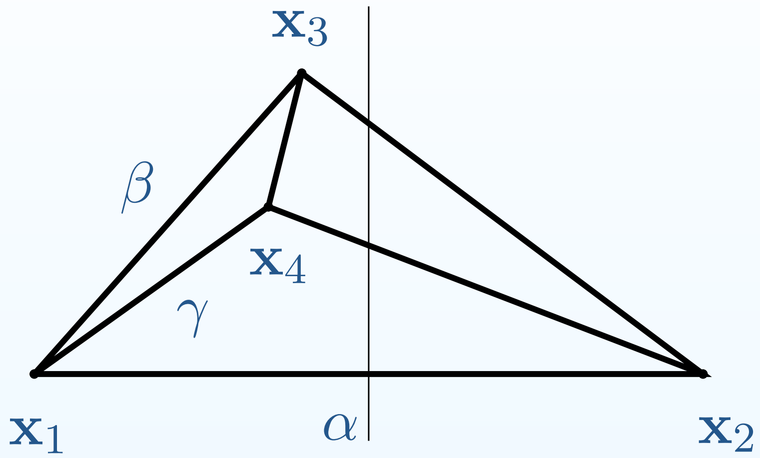
Consider cutting $\mathbb{R}^3$ with the plane that contains the midpoint of the edge $\mathbf{x}_1\mathbf{x}_2$ and is perpendicular to the vector $\mathbf{x}_1 - \mathbf{x}_2$. Then, there exist two cases:

(i) $\mathbf{x}_3$ and $\mathbf{x}_4$ belong to the same half-space,

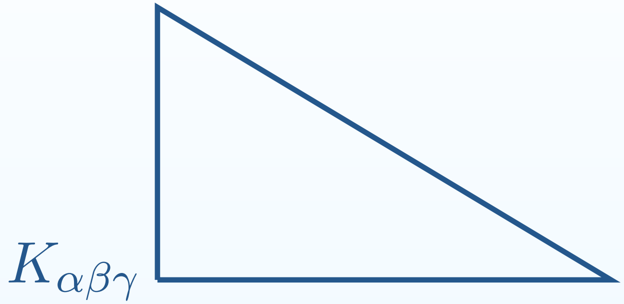
(ii) $\mathbf{x}_3$ and $\mathbf{x}_4$ belong to different half-spaces.

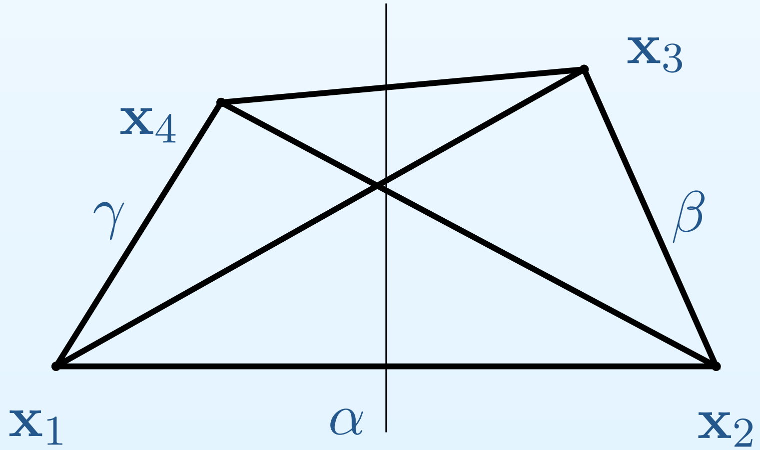
Let $\gamma := |\mathbf{x}_1 - \mathbf{x}_4|$.

The Standard Positions for Tetrahedrons

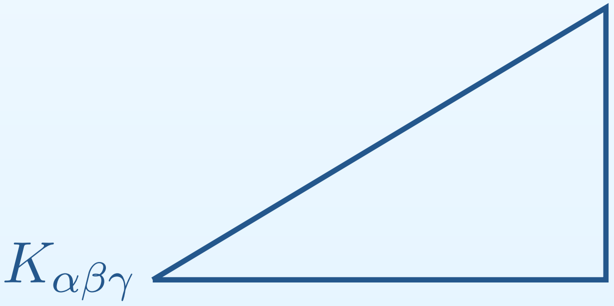


Case 1





Case 2



Under appropriate rotation, translation, and reflection operations, these situations can be written using the parameters

$$\begin{cases} 0 < \beta \leq \alpha, \quad 0 < \gamma, \quad s_1^2 + t_1^2 = 1,\ s_1 > 0,\ t_1 > 0, \quad \beta s_1 \leq \frac{\alpha}{2}, \\ s_{21}^2 + s_{22}^2 + t_2^2 = 1,\ t_2 > 0, \quad \gamma s_{21} \leq \frac{\alpha}{2}, \end{cases}$$

as

$$\mathbf{x}_1 = (0,0,0)^\top,\ \mathbf{x}_2 = (\alpha,0,0)^\top,\ \mathbf{x}_4 = (\gamma s_{21}, \gamma s_{22}, \gamma t_2)^\top,$$

$$\begin{cases} \mathbf{x}_3 = (\beta s_1, \beta t_1, 0)^\top & \text{for the case (i)} \\ \mathbf{x}_3 = (\alpha - \beta s_1, \beta t_1, 0)^\top & \text{for the case (ii)} \end{cases}.$$

We refer to the above coordinates as the **standard position** of a tetrahedron.

In the following, we sometimes write $h_B := \alpha$.
Let R_B be the circumradius of B.

General tetrahedrons

Define the matrices $\widehat{A}$, $\widetilde{A} \in GL(3,\mathbb{R})$ by

$$\widehat{A} := \begin{pmatrix} 1 & s_1 & s_{21} \\ 0 & t_1 & s_{22} \\ 0 & 0 & t_2 \end{pmatrix}, \quad \widetilde{A} := \begin{pmatrix} 1 & -s_1 & s_{21} \\ 0 & t_1 & s_{22} \\ 0 & 0 & t_2 \end{pmatrix}, \quad \begin{matrix} s_1^2 + t_1^2 = 1, \ t_1 > 0 \\ s_{21}^2 + s_{22}^2 + t_2^2 = 1, \ t_2 > 0 \end{matrix}$$

We then have

$$K = \widehat{A}(K_{\alpha\beta\gamma}) \text{ for case (i) or } K = \widetilde{A}(K_{\alpha\beta\gamma}) \text{ for case (ii).}$$

Note that $\widehat{A}$ and $\widetilde{A}$ can be decomposed as $\widehat{A} = X\widehat{Y}$ and $\widetilde{A} = X\widetilde{Y}$ with

$$X := \begin{pmatrix} 1 & 0 & s_{21} \\ 0 & 1 & s_{22} \\ 0 & 0 & t_2 \end{pmatrix}, \qquad \widehat{Y} := \begin{pmatrix} 1 & s_1 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \widetilde{Y} := \begin{pmatrix} 1 & -s_1 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

respectively.

The eigenvalues of $\widehat{Y}^\top \widehat{Y}$ and $\widetilde{Y}^\top \widetilde{Y}$ are 1, $1 \pm \mathbf{s}_1$ with $\mathbf{s}_1 := |s_1|$
The eigenvalues of $X^\top X$ are 1, $1 \pm \mathbf{s}_2$ with $\mathbf{s}_2 := \sqrt{s_{21}^2 + s_{22}^2}$.
Therefore, for $\mathbf{a} \in \mathbb{R}^3$, we have

$$(1 - \mathbf{s}_2)|\mathbf{a}|^2 \le |X\mathbf{a}|^2 \le (1 + \mathbf{s}_2)^2 |\mathbf{a}|^2,$$

$$(1 - \mathbf{s}_1)|\mathbf{a}|^2 \le |Z\mathbf{a}|^2 \le (1 + \mathbf{s}_1)|\mathbf{a}|^2, \qquad Z = \widehat{Y} \text{ or } Z = \widetilde{Y},$$

$$\prod_{i=1}^{2}(1 - \mathbf{s}_i)|\mathbf{a}|^2 \le |V\mathbf{a}|^2 \le \prod_{i=1}^{2}(1 + \mathbf{s}_i)|\mathbf{a}|^2, \qquad V = \widehat{A} \text{ or } V = \widetilde{A}.$$

Theorem 6 *Let K be an arbitrary tetrahedron at the standard position. Let k, m be integers with $k \ge 1$ and $0 \le m \le k$. Let p be taken as in the Squeezing Theorem according to k and m. Then, we have*

$$B_p^{m,k}(K) := \sup_{v \in \mathcal{T}_p^k} \frac{|v|_{m,p,K}}{|v|_{k+1,p,K}} \le C \frac{(\max\{\alpha, \beta, \gamma\})^{k+1-m}}{\prod_{i=1}^{2}(1 - \mathbf{s}_i)^{m/2}},$$

where $C = C(k, m, p)$ is a constant independent of K.

A geometric interpretation

Lemma 7 *Let K be a tetrahedron at the standard position. We then have*

$$\prod_{i=1}^{2}(1-\mathbf{s}_i)^{-1/2} \le C\frac{R_B R_P}{h_B \max\{\alpha,\beta,\gamma\}},$$

where C is a constant independent of K.

Corollary 8 *The following estimation holds;*

$$B_p^{m,k}(K) := \sup_{v\in\mathcal{T}_p^k} \frac{|v|_{m,p,K}}{|v|_{k+1,p,K}} \le C\left(\frac{R_B R_P}{h_B}\right)^m (\max\{\alpha,\beta,\gamma\})^{k+1-2m}.$$

Conclusions

- A new error estimation of Lagrange interpolation on tetrahedrons is obtained.
- In the new error estimation, the error of Lagrange interpolation is estimated in terms of the projected circumradius and the diameter of tetrahedrons. Geometric conditions are not imposed.
- The authors are not so sure if the projected circumradius is the **best** interpretation of the singular values of the linear transformation.
- Further consideration on the geometry of tetrahedrons is required.
- *Theoretically interesting problem*: extend the estimation to the case of d-simplex, $d \geq 4$.

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