Numerical verification of existence of homoclinic orbits in dynamical systems

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Introduction

We propose

- Numerical verification methods to prove existence of homoclinic orbits

in dynamical systems described by ODEs.

We treat an example problem with a saddle equilibrium in $\mathbb{R}^3$ of

$$\frac{d}{dt}\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x})$$

which has

- 1-dimensional stable manifold and 2-dimensional unstable manifold.

Introduction

Previous research:

D. Wilczak,

The Existence of Shilnikov Homoclinic Orbits in the Michelson System: A Computer Assisted Proof,

Foundations of Computational Mathematics,

6(4), 2006, pp. 495-535.

Introduction

Taking the assistance of reversibility, they reduce to problems for verifying connecting orbits of hyperbolic equilibria with

- 1-dimensional unstable manifold and 2-dimensional stable manifold.

In this case they can use a kind of intermediate value theorem with

- separating functions

to capture homoclinic and heteroclinic orbits.

In order to follow the unstable manifold, they use a topological tool so called

- covering relation.

Introduction

We will treat a problem which is hard to convert to have 1-dimensional unstable manifold because of numerical instability. In this case it is not easy to construct a separating function. Instead we use

- mapping degree and Brouwer coincidence theorem.

In order to specify the stable and unstable manifolds, our previous works on

- verified numerical methods of constructing Lyapunov functions

are available :

K. Matsue, T. Hiwaki and N. Yamamoto,
On the construction of Lyapunov functions with computer assistance,
arXiv preprint, arXiv:1604.05953

Our problem

Consider an autonomous system of ODEs :

$$\frac{d}{dt}\mathbf{x} = \mathbf{f}(\mathbf{x};\mathbf{p}), \quad \mathbf{x}, \mathbf{f} \in \mathbb{R}^3, \tag{1}$$

where $\mathbf{p} \in \mathbb{R}^2$ is a pair of parameters $\mathbf{p} = (a, b)$.

Suppose the following :

- For some parameter set, the system (1) may have a hyperbolic equilibrium with 2-dimensional unstable manifold such that a homoclinic orbit is expected to exit.
- For simplicity, the system (1) should have the same equilibrium $\boldsymbol{x}^*$ for any parameter set .

Our Aim

Our Aim is

- to prove existence of a homoclinic orbit ,

more precisely,

- to specify a narrow area which contains a parameter set attaining the existence of a homoclinic orbit.

Our Approach

Preparation :

- Find a parameter set to have a homoclinic-like flow, and compute the approximate flow.
- Then chose a small rectangular domain which contains the parameter set.
- And then construct Lyapunov functions for each parameter sets in the rectangle.

Since the rectangle is chosen to be small, it is often possible to find a fixed quadratic form to be Lyapunov function for all parameter sets in the rectangle.

Our Approach

In order to construct a Lyapunov function,

- Chose a quadratic form

$$(\boldsymbol{x} - \boldsymbol{x}^*)^T Y (\boldsymbol{x} - \boldsymbol{x}^*)$$

  by solving Lyapunov equation at the equilibrium.

Then specify

- a domain D_L including the equilibrium where the quadratic form is a Lyapunov function w.r.t. the equilibrium.

Our Approach

In order to do that, using verified numerics, we make sure that

- the cone condition holds within the domain D_L from negative definiteness of

$$Df(\boldsymbol{x})^T Y + Y Df(\boldsymbol{x}),$$

where $Df(\boldsymbol{x})$ denotes the Jacobi matrix of $\boldsymbol{f}(\boldsymbol{x};\boldsymbol{p})$ w.r.t. $\boldsymbol{x}$.

Again, since the rectangle is small, it is often possible to specify D_L to be a common subset of the domains of Lyapunov functions for all parameter sets in the rectangle.

Our Approach

This is our approach :

Consider a mapping consisting of two parts:

1. a continuous mapping from the rectangle of parameter sets to points on the unstable manifolds within D_L.
2. each flow from a point on the unstable manifold to a point on the 0-level set of Lyapunov function.

Then check

- the mapping degree,

and apply

- Brouwer coincidence theorem

in order to verify that there exists a pair of parameters in the rectangle which is mapped to the equilibrium on the 0-level set.

Our Approach

This approach is

- carried on by verified numerics

and then gives

- a proof of the existence of a homoclinic orbit corresponding to the pair of parameters.

Brouwer coincidence theorem

- B^2 : a unit circle in $\mathbb{R}^2$
- S^1 : the circumference of B^2
- F : a continuous mapping $B^2 \to B^2$ which satisfies $F(S^1) \subset S^1$

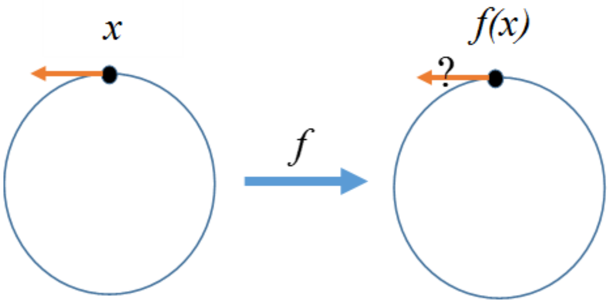
If the degree of the mapping $F(S^1)$ is not 0,
then for an arbitrary continuous mapping $G : B^2 \to B^2$,

- there exists a point $\boldsymbol{x} \in B^2$ s.t.

$$F(\boldsymbol{x}) = G(\boldsymbol{x}).$$

写像度

円周 S^1 について，$F : S^1 \to S^1$ となる連続写像 F を考える．点 x が S^1 上を正の向きに一周するとき，その像 $F(x)$ が S^1 上を何周するかを，符号まで考えて数えたときの整数を写像度と呼ぶ.



Interval Simplex Theorem

In order to check the degree

- It is not sufficient to compute the images of the mapping by verified computation.

IS Theorem is derived by the authors to give a way

- how to prove that the degree of a continuous mapping F is not 0 by interval arithmetic.

Interval Simplex Theorem

Consider mappings on the complex plane.

- Divide S^1 into Y_1, Y_2, Y_3 as follows.

$$Y_1 = \{\boldsymbol{z} \in S^1 | 0 \leq arg(\boldsymbol{z}) \leq 2\pi s_1\},$$
$$Y_2 = \{\boldsymbol{z} \in S^1 | 2\pi s_1 \leq arg(\boldsymbol{z}) \leq 2\pi s_2\},$$
$$Y_3 = \{\boldsymbol{z} \in S^1 | 2\pi s_2 \leq arg(\boldsymbol{z}) \leq 2\pi\},$$

  where $0 < s_1 < s_2 < 1$.

- Compute three sets V_i $(i = 1, 2, 3)$ which contains $F(Y_i)$ respectively.

Computation should be carried on by interval arithmetic and verified numerics.

Interval Simplex Theorem

The degree of F is not 0 if the following conditions hold.

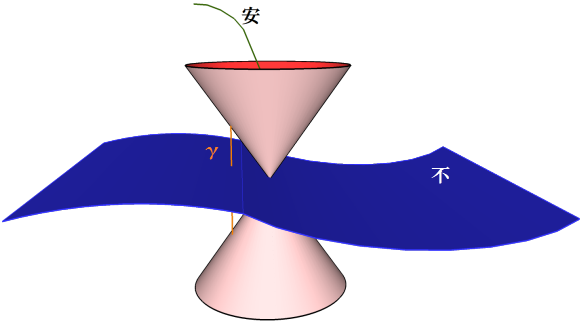
$$V_1 \cup V_2 \cup V_3 = S^1,$$
$$V_1 \cap V_2 \neq \phi,$$
$$V_2 \cap V_3 \neq \phi,$$
$$V_3 \cap V_1 \neq \phi,$$
$$V_1 \cap V_2 \cap V_3 = \phi.$$

ϕ : the empty set

Steps to verify homoclinic orbits

1. Take two parameters w.r.t. $\boldsymbol{f}(\boldsymbol{x})$, the right-hand side of given system of ODEs, and write them a, b and $\boldsymbol{p} = (a, b)$.
2. Take a rectangular domain $D_P = \{(a, b) | a \in [\underline{a}, \overline{a}], b \in [\underline{b}, \overline{b}]\}$. Note that we use ∂D_P, the boundary of D_P, instead of S^1 taking an isomorphism from S^1 to ∂D_P.
3. Construct a Lyapunov function $L(\boldsymbol{x})$ and specify its domain D_L by verified computation. Draw a cone as the 0-level set of the Lyapunov function.
4. Take a line segment $\gamma \subset D_L$ whose end points are settled on the upper and the lower part of the cone.

5. Specify small intervals which include the intersection points of γ and the unstable manifolds. It is proved that these points are unique and continuous w.r.t. parameter $\boldsymbol{p} \in D_P$ by virtue of the cone condition.
6. Taking the above intervals as initial interval values, calculate the flows by e.g. Lohner method. Verify that the flows once go out of D_L and come into D_L again, and stay within the body of the cone.



The flows within the body of the cone should

- run out through the 0-level set $L^{-1}(0)$,

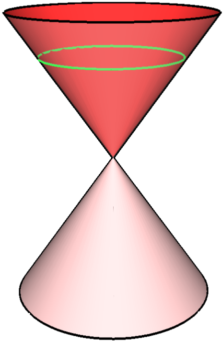
  or

- run toward the equilibrium $\boldsymbol{x}^*$.

This means that we can construct a continuous mapping

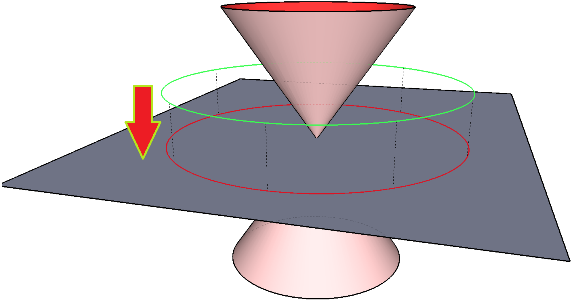
$$F : D_P \to L^{-1}(0)$$

7. Verify that the parameter sets on ∂D have their images on $L^{-1}(0)$, which surrounds the axis of the cone.
8. Verify the degree of F is not 0 by our IS theorem.



In practical verification we use the following steps.

7. Consider a plane Γ which includes the equilibrium. Compute the flows w.r.t. $\boldsymbol{p} \in \partial D_P$ up to a certain time. Verify that the set of the end points of the flows surrounds the cone body. Then project the set onto Γ and let $\hat{S}$ denotes its image.
8. Check the degree of the mapping $\hat{F}$ from ∂D_P to $\hat{S}$, and verify that it is not 0.



Notes:

- Lyapunov functions enable us to capture a region in D_L where the unstable manifolds exist.
- In order to guarantee the continuity of the mapping F, a technique on substitution integration so called Lyapunov Tracing is applied.
- We have to check other conditions on the boundary of D_L.
- We use more complicated arguments to construct the mapping F in actual, e.g. applying a shrinking method to $\hat{S}$.
- The degrees of F and $\hat{F}$ coincide as long as the inclination of the axis of the cone is not too much.
- With some additional conditions, $\hat{F}$ allows us to extend our method to the cases that equilibriums and Lyapunov functions are varied w.r.t. the parameters.

Example

We use Intlab for verified computation on our examples.

- System of ODEs

$$\begin{cases} \frac{dx}{dt} = -y - z \\ \frac{dy}{dt} = x + ay \\ \frac{dz}{dt} = bx - cz + xz \end{cases}$$

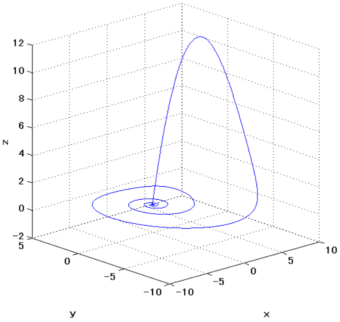


Figure: Approximation to homoclinic orbit

Example

- The origin is an equilibrium for any coefficients.
- The system has 1-dimensional stable manifold and 2-dimensional unstable manifold..
- It is observed that the system may have a homoclinic orbit when $a = 0.38, b = 0.30, c = 4.82.$

西浦廉政 (Y. Nishiura) "非平衡ダイナミクスの数理"，岩波書店，2009

Parameters

- We take the coefficients a and b as parameters since the solution has more sensitivity to those than the coefficient c.
- The rectangular domain of parameter sets : $D_P = [0.3797, 0.3804] \times [0.2993, 0.3008]$.

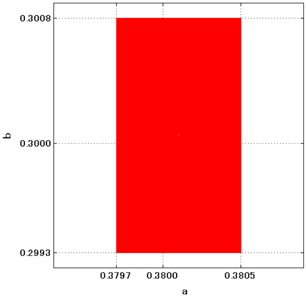


Figure: Parameter sets

Construction of Lyapunov function

- We construct a Lyapunov function L w.r.t. the equilibrium $\boldsymbol{x}^* = (0, 0, 0)$.
  Using Jacobi matrix of the right-hand side at $\boldsymbol{x}^*$ for $a = 0.38, b = 0.30$,

$$
\begin{aligned}
L(\boldsymbol{x}; \boldsymbol{\lambda}) &= (\boldsymbol{x} - \boldsymbol{x}^*)^T Y (\boldsymbol{x} - \boldsymbol{x}^*) \\
&= \boldsymbol{x}^T \begin{pmatrix} -1.0671 & -0.2324 & 0.1424 \\ -0.2324 & -1.0584 & -0.0071 \\ 0.1424 & -0.0071 & 1.0257 \end{pmatrix} \boldsymbol{x}
\end{aligned}
$$

  is derived as a candidate for Lyapunov function.

Specify the domain of Lyapunov function

It is verified that the function L is a Lyapunov function for any parameters in D_P within

$$[x] = [-1.0, 1.0]$$
$$[y] = [-1.0, 1.0]$$
$$[z] = [-0.5, 1.0]$$

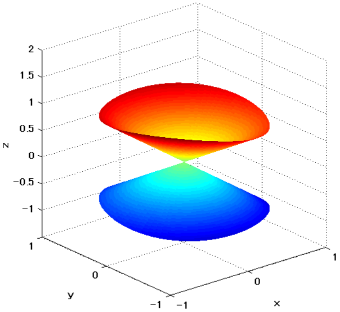


Figure: The cone $L = 0$

Specify the points on the unstable manifolds

The line segment γ : the end points are settled on the upper and the lower part of the cone.

- We take γ as a segment parallel to the z axis including a point on the approximate homoclinic orbit.
- Using verified computation, we have an interval part of γ which includes the intersection points between γ and the unstable manifolds:

$$(1.0 \times 10^{-5}, 1.0 \times 10^{-4}, [-5.8 \times 10^{-7}, -5.6 \times 10^{-7}])$$

for any parameter within D_P.

Integration

- Taking the above interval part as an initial interval value, we integrate the trajectories and have verified that all trajectories for D_P come into the cone body until a certain time.
- We integrate the trajectories for the parameters on the boundary ∂D_P for longer time.

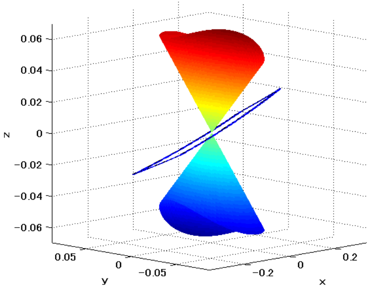


Figure: ring surrounding the cone

- For parameters over ∂D_P, a ring surrounding the cone is observed.

Projection to Γ

- We take the xy plane as the plane Γ.
- The ring is projected on Γ and $\hat{S}$ denotes its image.
- In this case the projection maps any ring surrounding the cone to a ring surrounding the equilibrium. Therefore the degrees of F and $\hat{F}$ w.r.t. the the equilibrium are the same.

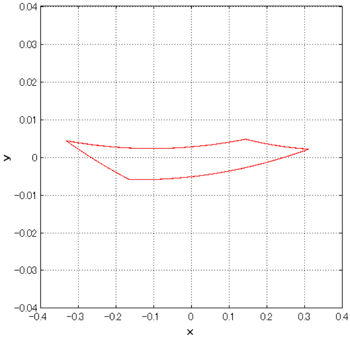


Figure: $\hat{S}$

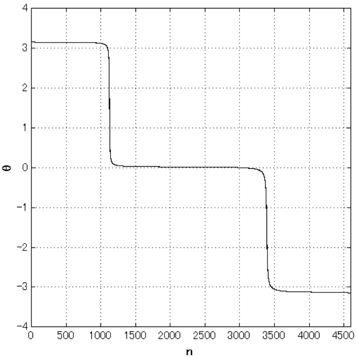
Verify the degree

- We have applied the IS theorem to this example and verified that the degree of $\hat{F}$ is not 0.
- Consequently we have verified the existence of a homoclinic orbit for a certain parameter set within the domain D_P.

Verify the degree

Besides, we illustrate how $\hat{F}$ maps the points on the boundary which shows the degree is 1.

- Chose 5000 points on ∂D_P.
- Calculate the images on $\hat{S}$ and put out the angles θ_i of the position vectors of the images to positive direction of the x axis.
- Plotting θ_i, we can observe that the angles change from π to $-\pi$ when $\boldsymbol{p} \in \partial D_P$ moves clockwise.



Concluding Remarks

- We present a numerical verification method for existence of homoclinic orbits with 2-dimensional unstable manifolds.
- We adopt Brouwer coincidence theorem and mapping degree.
- A new theorem is derived in order to check the degree by interval arithmetic.
- This method can be applied to other cases, e.g. heterocrinic orbits or higher dimensional problems, as long as they have 2-dimensional unstable manifolds.
- To treat higher dimensional unstable manifolds, we have to derive IS theorem for S^n, $n > 1$. This would be our future work.