Hamiltonian S^1-spaces with large minimal Chern number

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Minimal Chern number

(M, ω): compact symplectic manifold of dimension $2n$.

k:=**minimal Chern number of** (M, ω)

- If there exists $S \in \pi_2(M)$ s.t. $c_1(S) > 0$ then

$$k \in \mathbb{Z}_{>0}: \quad \langle c_1, \pi_2(M) \rangle = k\mathbb{Z}\,;$$

- otherwise $k = \infty$

Minimal Chern number

(M, ω): compact symplectic manifold of dimension $2n$.

Questions:

Suppose (M, ω) is *positive monotone* i.e.

$$c_1 = \lambda[\omega] \quad \lambda > 0$$

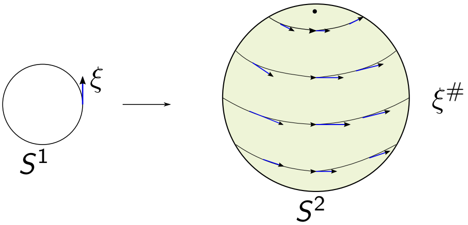
and $k < \infty$.

- $k \leq n + 1$?
- What about $k = n + 1$? $M \simeq \mathbb{C}P^n$?

Symplectic S^1-actions

$S^1 \curvearrowright (M, \omega)$

$\xi^{\#}$: vector field associated to the flow of symplectomorphisms



Symplectic S^1-actions

Flow of symplectomorphisms $\Longrightarrow$ $\iota_{\xi^\#}\omega$ is **closed**.

- *Hamiltonian* : $\iota_{\xi^\#}\omega$ is **exact**, i.e. $\exists\ \psi \in C^\infty(M)$ s.t.

$$\boxed{\iota_{\xi^\#}\omega = d\psi}$$

  $\psi\colon M \to \mathbb{R}$: *moment map*.
- Otherwise *non-Hamiltonian*.

Hamiltonian S^1-spaces

Hamiltonain $S^1 \curvearrowright (M, \omega)$, $\psi \colon M \to \mathbb{R}$

M^{S^1}:= **set of fixed points**

Hamiltonian $S^1 \curvearrowright (M, \omega) \implies M^{S^1} \neq \emptyset$

$$\boxed{\iota_{\xi^\#}\omega = d\psi}$$

Assume M^{S^1} is **discrete**

We call the triple (M, ω, S^1),

- with $S^1 \curvearrowright (M, \omega)$ Hamiltonian,
- M^{S^1} discrete,

a **Hamiltonian S^1-space**.

Example of Hamiltonian S^1-space

$$(\mathbb{C}P^n, \omega_{FS}, S^1)$$

S^1 action:

$$S^1 \ni \lambda * [z_0 : z_1 : \ldots : z_n] = [z_0 : \lambda^{a_1} z_1 : \lambda^{a_2} z_2 : \ldots : \lambda^{a_n} z_n]$$

with $a_1 < a_2 < \ldots < a_n$, $a_i \in \mathbb{Z} \setminus \{0\}$ for all i.
Fixed points:

$$[1 : 0 : \ldots : 0], \quad [0 : 1 : \ldots : 0], \quad \ldots, \quad [0 : 0 : \ldots : 1]$$

$c_1 = (n+1)x$, x gen. of $H^2(\mathbb{C}P^n; \mathbb{Z})$

$$k = n + 1$$

Results:

(M, ω, S^1) Hamiltonian S^1-space.

1. $1 \leq k \leq n+1$

   S., *"On the Chern numbers and the Hilbert polynomial of an almost complex manifold with a circle action"*, Commun. Contemp. Math., **19** (2017).

2. If $k = n+1$, $c_1 = \lambda[\omega]$ (+ two technical hypotheses)

   (a) $\chi(M) = n+1$

   Godinho, von Heymann, S., *"12, 24 and beyond"*, Adv. Math., **319** (2017).

   (b) M is homotopy equivalent to $\mathbb{C}P^n$.

   Charton, *"Hamiltonian manifolds with high index"*, Master thesis. University of Cologne, 2017.

$1 \leq k \leq n+1$

Index k_0 of (M, ω): *largest integer* s.t. (modulo torsion)

$$\boxed{c_1 = k_0 \, \eta_0}$$

for some non-zero $\eta_0 \in H^2(M; \mathbb{Z})$.

- $k_0 \geq 0$
- $k_0 = 0$ exactly if c_1 is torsion.

$1 \leq k \leq n+1$

(M, ω, S^1) Hamiltonian S^1-space:

Facts:

1. c_1 is not torsion (Hattori '84, Tolman '10)
2. M is simply connected

1. $\implies$ $k_0 > 0$
2. $\implies$ index $k_0 =$ minimal Chern number k.

So

$$\boxed{1 \leq k}$$

$1 \leq k \leq n+1$

Idea of the proof of upper bound:

Find a polynomial $H(z)$ s.t.

- $0 < \deg(H) \leq n$
- H is zero at $-1, -2, \ldots, -k+1$

$\implies k-1 \leq \deg(H) \leq n$.

$1 \leq k \leq n+1$

Consider the polynomial $H(z)$ such that

$$\boxed{H(m) = \int_M e^{m\,\eta_0}\,\mathcal{T}(M) \quad \text{for all} \quad m \in \mathbb{Z}}$$

where:

- $e^{m\,\eta_0} = \sum_{j\geq 0} \frac{(m\,\eta_0)^j}{j!}$
  Chern character of the Line bundle $\mathbb{L}^m$ whose first Chern class is $m\,\eta_0$ $(c_1 = k\eta_0)$.

- $\mathcal{T}(M)$ is the total Todd class of M, i.e. $\mathcal{T}(M) = \sum_j T_j$.

T_j: j-th Todd polynomial, e.g.

$T_0 = 1, \quad T_1 = \frac{c_1}{2}, \quad T_2 = \frac{c_1^2 + c_2}{12}, \quad T_3 = \frac{c_1 c_2}{24}, \ldots$

Facts about H:

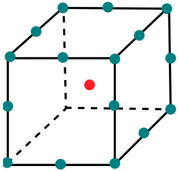
- $\deg(H) \leq n$
- $H(0) = 1$ for (M, ω, S^1) with M connected (Feldman '01)
- If $k \geq 2$ then $H(-1) = H(-2) = \ldots = H(-k+1) = 0$ (S. '17)

Sketch of the proof of the **zeros of** $H(z)$:

1. $H(m) =$ topological index of the bundle $\mathbb{L}^m$, for all $m \in \mathbb{Z}$, where $c_1(\mathbb{L}) = \eta_0$ and $c_1 = k\,\eta_0$ (Atiyah-Singer Index Theorem);
2. $\mathbb{L}^m$ admits an equivariant extension $\mathbb{L}^m_{S^1}$ for every $m \in \mathbb{Z}$ (Hattori);
3. the equivariant index $P_m(t)$ of $\mathbb{L}^m_{S^1}$ is in $\mathbb{Z}[t, t^{-1}]$, and for $t = 1$ gives $H(m)$;
4. the equivariant index can be computed using a localization formula in equivariant K-theory;
5. using this formula, compute the limits of $P_m(t)$ for $t \to \infty$ and $t \to 0$, and realize that these limits are both zero for all $m \in \{-1, \ldots, -k+1\}$; hence $P_m(t) \equiv 0 \Longrightarrow$ $H(m) = 0$ for all $m \in \{-1, \ldots, -k+1\}$.

Relations between Betti numbers and k

The **toric one-skeleton** of a symplectic toric manifold (M, ω, μ) is the family of smoothly embedded, symplectic, invariant 2-spheres corresponding to the edge set of $\mu(M) = \Delta$



$e \longleftrightarrow S^2_e := \mu^{-1}(e).$

$$\textbf{Fact: } c_{n-1} = \mathrm{PD}\left[\bigcup_{e \in E} S^2_e\right].$$

Relations between Betti numbers and k

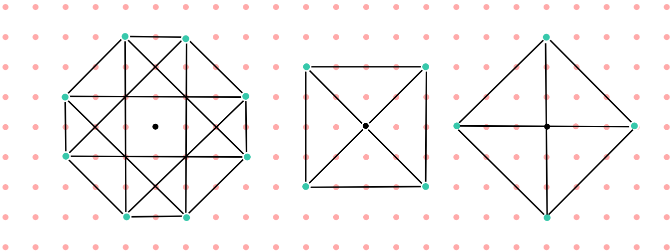
Many Hamiltonian S^1-spaces admit a *toric one-skeleton*.

- All Hamiltonian S^1-spaces admit a *multigraph* $\Gamma = (V, E)$ that describes the action;
- A toric one-skeleton (when it exists) is the union of finitely many smoothly embedded, symplectic, invariant 2-spheres, each one "associated" to exactly one edge of Γ.

Fact: If the toric one-skeleton exists:

$$c_{n-1} = \mathrm{PD}\left[\bigcup_{e \in E} S_e^2\right].$$

E.g. Hamiltonian GKM (*Goresky-Kottwitz-MacPherson*) spaces (for instance, coadjoint orbits of compact simple Lie groups).
$\mu(\cup_{e\in E} S^2_e)$ is a graph in $\mathrm{Lie}(\mathbb{T})^*$:



Relations between Betti numbers and k

Theorem (Godinho-S. '12, G.-S.- von Heymann '17)

Let (M, ω, μ) be a Hamiltonian S^1-space of dimension $2n$ with toric one-skeleton $\{S^2_e\}_{e\in E}$. Let $b_i(M)$ be the Betti numbers of M, and $\chi(M)$ its Euler characteristic.
Then $\sum_{e\in E} c_1[S^2_e]$ **only depends on the Betti numbers.**
More precisely:

- If n is *even*

$$\sum_{e\in E} c_1[S^2_e] + \frac{n}{2}\,\chi(M) = 12 \sum_{k=1}^{\frac{n}{2}} \left[k^2 b_{n-2k}(M)\right]$$

- If n is *odd*

$$\sum_{e\in E} c_1[S^2_e] + \left(\frac{n-3}{2}\right)\chi(M) = 12 \sum_{k=1}^{\frac{n-1}{2}} \left[k(k+1) b_{n-1-2k}(M)\right]$$

Special cases:

Under the same hypotheses:

- If $n = 2$ then
$$\sum_{e\in E} c_1[S^2_e] + \chi(M) = \mathbf{12}\,.$$
- If $n = 3$ then
$$\sum_{e\in E} c_1[S^2_e] = \mathbf{24}\,.$$

Sketch of the Proof:

- Since $c_{n-1} = \mathrm{PD}\left[\cup_{e \in E} S^2_e\right]$,

$$\sum_{e \in E} c_1[S^2_e] = c_1 c_{n-1}[M].$$

- $c_1 c_{n-1}[M]$ only depends on the Betti numbers ("Rigidity of Hirzebruch genus") (Godinho-S. '12).

Relations between Betti numbers and k

Hamiltonian S^1-spaces which are *"positive"*:

$$c_1[S^2_e] > 0 \quad \text{for all} \quad e \in E\,.$$

- **E.g.** : (M, ω, S^1) *monotone*, i.e. $c_1 = \lambda[\omega]$, $\lambda \in \mathbb{R}$ (For Hamiltonian S^1-spaces $\lambda > 0$).

$c_1[S^2_e] - k$ is a non-negative multiple of k, hence

$\sum_{e \in E} c_1[S^2_e] - k|E|$ is a non-negative multiple of k.

Both $\sum_{e \in E} c_1[S^2_e]$ and $|E|$ depend on the Betti numbers.

Relations between Betti numbers and k

(M, ω, S^1) *positive* Hamiltonian S^1-space with a toric one-skeleton and minimal Chern number k:

k	$n = 2$ b_2	$n = 3$ b_2	$n = 4$ (b_2, b_4)	$n = 5$ (b_2, b_4)
1	$b_2 \leq 4$	$b_2 \leq 7$		
2	2	$b_2 \leq 3$		
3	1	1	$(1, 2), (2, 3), (3, 1), (4, 2), (6, 1)$	
4		1	$(1, 2)$	
5			$(1, 1)$	$(1, 1), (6, 1)$
6				$(1, 1)$

Table: List of allowed values of b_2 and b_4 for $2 \leq n \leq 5$.

Relations between Betti numbers and k

Unimodality of even Betti numbers:
The sequence of even Betti numbers of (M, ω, S^1) is *unimodal* if:

$$b_0 \leq b_2 \leq \ldots \leq b_{2\lfloor \frac{n}{2} \rfloor}$$

Theorem (Cho '16)

The sequence of even Betti numbers of a Hamiltonian S^1-space admitting an *index-increasing* moment map is unimodal.

E.g. Transversality of unstable and stable manifolds of moment map $\implies$ index increasing.

$k = n+1 \implies \chi(M) = n+1$

Theorem (Godinho, von Heymann, S. '17)

Suppose (M, ω, S^1) is a Hamiltonian S^1-space s.t.

- it admits a *toric one-skeleton* (e.g. toric action, GKM, ...);
- it is *positive* , i.e. $c_1[S^2_e] > 0$ on all spheres of the toric one-skeleton (e.g. $c_1 = \lambda[\omega]$);
- the sequence of even Betti numbers is *unimodal* (e.g. transversality...).

If $k = n+1$, then $b_{2i}(M) = 1$ for all $i = 0, \ldots, n$ and $b_{odd}(M) = 0$, hence $\chi(M) = n+1$.

Sketch of the proof:

- $\sum_{e \in E} c_1[S^2_e] - k|E|$ is a non-negative multiple of k
- $\sum_{e \in E} c_1[S^2_e]$ and $|E|$ can be expressed as linear combinations of (even) Betti numbers.

From $\chi(M) = n+1$ to $M \simeq \mathbb{C}P^n$

Local form of the action around $p \in M^{S^1}$: ***Weights*** *at p*

If $p \in M^{S^1}$ then $S^1 \curvearrowright T_pM \simeq \mathbb{C}^n$:

$$\lambda \cdot (z_1, \ldots, z_n) = (\lambda^{w_{1p}} z_1, \ldots, \lambda^{w_{np}} z_n)$$

$w_{1p}, \ldots, w_{np} \in \mathbb{Z}$ are the **weights** of the S^1 action at p.

Example: $(\mathbb{C}P^n, \omega_{FS}, S^1)$

$$S^1 \ni \lambda * [z_0 : z_1 : \ldots : z_n] = [z_0 : \lambda^{a_1} z_1 : \lambda^{a_2} z_2 : \ldots : \lambda^{a_n} z_n]$$

with $a_1 < a_2 < \ldots < a_n$, $a_i \in \mathbb{Z} \setminus \{0\}$ for all i.

Weights at $p_0 := [1 : 0 : \ldots : 0]$: $a_1, a_2, \ldots, a_n$
$p_1 := [0 : 1 : \ldots : 0]$: $-a_1, a_2 - a_1, \ldots, a_n - a_1$
$\ldots$
$p_n := [0 : 0 : \ldots : 1]$: $-a_n, a_1 - a_n, \ldots, a_{n-1} - a_n$.

From $\chi(M) = n+1$ to $M \simeq \mathbb{C}P^n$

Hattori '84 $\Longrightarrow$

Theorem

(M, ω, S^1) Hamiltonian S^1-space with $\chi(M) = k = n+1$. Let $p_0, \dots, p_n$ be the fixed points.
Then $\exists\ a_1, \dots, a_n \in \mathbb{Z}$ s.t. the weights at $p_0, \dots, p_n$ are those of $\mathbb{C}P^n$ with "standard S^1-action".

The S^1-action on $TM|_{M^{S^1}}$ coincides with the standard S^1-action on $\mathbb{C}P^n$.

From $\chi(M) = n+1$ to $M \simeq \mathbb{C}P^n$

Hamiltonian S^1-space (M, ω, S^1), $\quad \chi(M) = n + 1 = |M^{S^1}|$

S^1-action is Hamiltonian $\implies H^{2i}(M;\mathbb{Z}) = \mathbb{Z}$ for all $i = 0, \dots, n$.
Cohomology ring structure?

$$c_1 = \lambda[\omega], \quad \lambda > 0 \implies c_1^i \neq 0 \in H^{2i}(M;\mathbb{Z}) \;\; i = 0, \dots, n \implies$$

$$\exists \quad N_i \in \mathbb{Z} \setminus \{0\} \quad \text{s.t.} \quad \frac{c_1^i}{N_i} \text{ is a generator of } H^{2i}(M;\mathbb{Z}), \quad i = 0, \dots, n$$

Theorem (Tolman '10)

Hamiltonian S^1-space (M, ω, S^1) with $\chi(M) = n + 1$. Then $H^*(M;\mathbb{Z})$ is determined by the weights at the fixed points (N_i's are determined by the weights).

Hence: if the weights agree with those of $\mathbb{C}P^n$, the cohomology ring does as well.

From $\chi(M) = n + 1$ to $M \simeq \mathbb{C}P^n$

Theorem (Charton '17)

Suppose (M, ω, S^1) is a Hamiltonian S^1-space s.t.

- it admits a *toric one-skeleton* (e.g. toric action, GKM, ...);
- it is *positive* , i.e. $c_1[S^2_e] > 0$ on all spheres of the toric one-skeleton (e.g. $c_1 = \lambda[\omega]$);
- the sequence of even Betti numbers is *unimodal* (e.g. transversality of stable and unstable manifolds of m.m. ...).

If $k = n + 1$, then M is homotopy equivalent to $\mathbb{C}P^n$.

From $\chi(M) = n + 1$ to $M \simeq \mathbb{C}P^n$

Sketch of the proof:

- (Godinho, von Heymann, S. '17) $\implies$ $\chi(M) = n + 1$;
- (Hattori '84, Tolman '10) $\implies$ $H^*(M; \mathbb{Z}) \simeq H^*(\mathbb{C}P^n; \mathbb{Z})$;
- M homotopy equivalent to a CW-complex X with one cell in dimension $0, 2, \ldots, 2n$, $\pi_1(M) = \pi_1(\mathbb{C}P^n) = 0$;
- Construct a map $f \colon X \to \mathbb{C}P^n$ s.t. $f_* \colon H_i(X) \to H_i(\mathbb{C}P^n)$ is an isomorphism for all i;
- (Corollary of Hurewicz) $\implies$ f is a homotopy equivalence.

Thank you!