Persistence modules with operators

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GEOMETRY (pseudo-metric space) $\rightsquigarrow$ ALGEBRA (persistence module) $\rightsquigarrow$ COMBINATORICS (barcode)

Pseudo-metric space	**Persistence module**
(Finite metric spaces, d_{GH})	Homology of Rips complex
X closed manifold $(C^0(X), \|\cdot\|_\infty)$	(Morse) homology of sublevel sets $H_*(\{f < t\})$
M monotone symplectic $(\widetilde{\mathrm{Ham}}(M), \tilde{d}_H)$	Filtered Floer homology $HF_*^t(\tilde{\phi}), \tilde{\phi} \in \widetilde{\mathrm{Ham}}(M)$

Key property - The above correspondence is Lipschitz with respect to pseudo-metric d_{bottle} on the space of barcodes.

Introduction

From the barcode one may read some previously known invariants such as:

- Critical values of a Morse function
- Homological min-max values of a Morse function
- Spectral invariants (Viterbo, Schwarz, Oh)
- Boundary depth (Usher)

Homology comes with product structure. In Morse case this is intersection product and in Floer case this is quantum or pair-of-pants product.

Goal - Take homological product into account and obtain finer invariants. This leads to the concept of a persistence module with an operator.

Barcodes

Barcode - A multiset $\mathcal{B} = \{I_j\}_{j \in \mathcal{J}}$ of intervals $I_j \subset \mathbb{R}$.

Distance

$\mathcal{B}$ and $\mathcal{B}'$ are ε-*matched* if after erasing some bars of length $< 2\varepsilon$ the rest are in bijection up to an error of ε on the endpoints.

Bottleneck distance is given by

$$d_{bottle}(\mathcal{B}, \mathcal{B}') = \inf\{\varepsilon \mid \mathcal{B}, \mathcal{B}' \text{ are } \varepsilon\text{-matched}\}.$$

Examples:

$$d_{bottle}(\{(0,2], [0,1]\}, \{(0,2.1)\}) = \frac{1}{2}$$

$$d_{bottle}(\{(0,2], [0,1]\}, \{(0,+\infty)\}) = +\infty$$

Persistence module - A family of finite dimensional vector spaces $V^t,\ t \in \mathbb{R}$ and comparison maps $\pi_{s,t} : V^s \to V^t$ for $s \leq t$, which satisfy

$$\pi_{s,t} \circ \pi_{r,s} = \pi_{r,t},\ \pi_{t,t} = \mathbf{1}_{V^t}.$$

Morphism of persistence modules - A family of linear maps $A_t : V^t \to W^t$, $t \in \mathbb{R}$ commuting with comparison maps. We have well defined $\ker A$ and $\operatorname{im} A$.

Main examples

Filtered homology - For a Morse function f, set $V^t = H_*(\{f < t\})$, $\pi_{s,t}$ are induced by inclusions of sublevel sets.

Floer persistence module - (M, ω) monotone symplectic manifold, meaning $\omega|_{\pi_2(M)} = \kappa \cdot c_1(TM)|_{\pi_2(M)}$, $\kappa > 0$.

For a fixed degree $k \in \mathbb{Z}$ and a non-degenerate, time-dependent Hamiltonian H, set $V^t = HF_k(\{\mathcal{A}_H < t\})$. Comparison maps $\pi_{s,t}$ are induced by inclusions of generators of chain complex.

Resulting persistence module depends only on the class $[\phi_1^H]$ inside the universal cover of $\mathrm{Ham}(M)$. We will denote it by $HF_k^t(\tilde{\phi})_{pt}$ for $\tilde{\phi} \in \widetilde{\mathrm{Ham}}(M)$.

Remark: There is an analogue persistence module $HF_k^t(\tilde{\phi})_\alpha$ whose underlying chain complex is generated by periodic orbits in a fixed homotopy class α. In this case we will assume that α is toroidally monotone class.

Structure theorem

Interval module - $\mathbb{K}(I)$ is given by

$$\mathbb{K}(I)^t = \begin{cases} \mathbb{K}, & \text{for } t \in I \\ 0, & \text{otherwise} \end{cases} \qquad \pi_{s,t} = \begin{cases} \mathbf{1}_{\mathbb{K}}, & \text{for } s,t \in I \\ 0, & \text{otherwise} \end{cases}$$

The structure theorem (A. Zomorodian, G. Carlsson - 2005.)

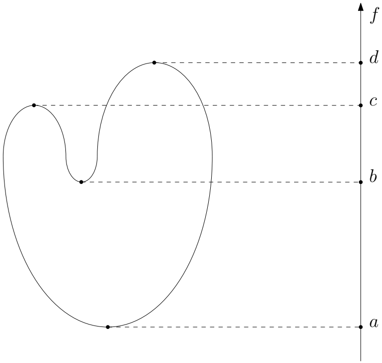
For every persistence module V there is a unique barcode $\mathcal{B}(V)$ s.t.

$$V \cong \bigoplus_{I \in \mathcal{B}(V)} (\mathbb{K}(I))^{m_I},$$

where m_I stands for the multiplicity (number of copies) of I in $\mathcal{B}(V)$.

Structure theorem - an example

$f : S^1 \to \mathbb{R}$ is a hight function on deformed circle given by:



$$\mathcal{B}_1(f) = \{(d, +\infty)\}, \mathcal{B}_0(f) = \{(a, +\infty), (b, c]\}.$$

The isometry theorem

Interleaving - V and W are δ-interleaved if there exist morphisms $F : V \to W[\delta]$ and $G : W \to V[\delta]$ such that

$$G_{t+\delta} \circ F_t = \pi^V_{t,t+2\delta}, \quad F_{t+\delta} \circ G_t = \pi^W_{t,t+2\delta}.$$

Interleaving distance

$$d_{inter}(V, W) = \inf\{\delta > 0 \mid V, W \text{ are } \delta - \text{interleaved}\}.$$

The isometry theorem (D. Cohen-Steiner, H. Edelsbrunner, J. Harer 2007. and M. Lesnick 2011.)

$$d_{inter}(V, W) = d_{bottle}(\mathcal{B}(V), \mathcal{B}(W))$$

Strategy - Construct interleavings, but analyse barcodes.

Stability

Morse case

f, g - Morse functions on the same manifold, $\|f - g\|_\infty = C$.

$V^t(f)$, $V^t(g)$ - associated filtered homologies.

We have

$$\{f < t\} \subset \{g < t + C\} \subset \{f < t + 2C\}.$$

These inclusions induce C-interleaving between $V^t(f)$, $V^t(g)$ which gives

$$\|f - g\|_\infty \geq d_{inter}(V(f), V(g)) = d_{bottle}(\mathcal{B}(V(f)), \mathcal{B}(V(g))).$$

Stability

Floer case

Let α be toroidally monotone class of free loops in M, $[\phi_t^F], [\psi_t^G] \in \widetilde{\mathrm{Ham}}(M)$ be generated by non-degenerate Hamiltonians.

Continuation maps induce interleavings as follows:

$$HF_*^t(F)_\alpha \xrightarrow{C(F,G)} HF_*^{t+\mathcal{E}^+(G-F)}(G)_\alpha \xrightarrow{C(G,F)} HF_*^{t+\mathcal{E}(G-F)}(F)_\alpha,$$

where

$$\mathcal{E}(G-F) = \int_0^1 (\max_{x\in M}(G_t(x) - F_t(x)) - \min_{x\in M}(G_t(x) - F_t(x))).$$

Taking infimum of $\mathcal{E}(G-F)$ over F, G generating $\tilde{\phi}, \tilde{\psi}$ gives us

$$\tilde{d}_H(\tilde{\phi}, \tilde{\psi}) \geq d_{inter}(HF_k^t(\tilde{\phi})_\alpha, HF_k^t(\tilde{\psi})_\alpha) = d_{bottle}(\mathcal{B}(\tilde{\phi})_\alpha, \mathcal{B}(\tilde{\psi})_\alpha).$$

Persistence modules with operators

Persistence module with an operator - A pair (V, A), where $V = (V^*, \pi_*)$ is a persistence module and

$$A_t : V^t \to V^{t+c_A}, \; t \in \mathbb{R}$$

is a morphism from V to the shifted module $V[c_A]$. In the case of graded vector spaces we also allow A to shift degree by a fixed constant.

Morphism of persistence modules with operators - Same as before plus commutes with operators.

Examples

$\mathbb{Z}_p$-action

A persistence module V with an operator

$$A_t : V^t \to V^t, \quad A^p = \mathbf{1}_V$$

is called a $\mathbb{Z}_p$ persistence module.

Intersection product

Let $\dim M = n$. Fixing a homology class $e \in H_r(M)$ we have

$$(e\cap) : H_k^t(\{f < t\}) \to H_{k+r-n}^t(\{f < t\}),$$

given by intersecting cycles inside sublevel set $\{f < t\}$ by a cycle representing e.
In other words

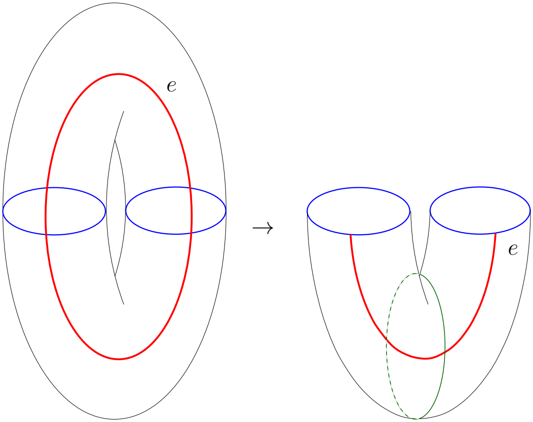
$$(e\cap) : V^t \to V^t,$$

where $V^t = H_*(\{f < t\})$.

Examples

For intersection with a circle on the torus we have $\dim M = 2$, $e \in H_1(M)$ and

$$(e\cap) : H_k^t(\{f < t\}) \to H_{k-1}^t(\{f < t\}).$$



Quantum product

M - monotone symplectic manifold, c_M - minimal Chern number. Introduce

$$\Lambda_{\mathbb{K}} = \left\{ \sum_{n\in\mathbb{Z}} a_n q^n \;\middle|\; a_n \in \mathbb{K},\ (\exists n_0 \in \mathbb{N})\ a_n = 0 \text{ for } n \geq n_0 \right\}.$$

Quantum homology of M over $\mathbb{K}$ - $QH(M) = H_*(M,\mathbb{K}) \otimes_{\mathbb{K}} \Lambda_{\mathbb{K}}$

Grading is given by declaring $\deg q = 2c_M$.

We can define natural valuation $\nu : QH(M) \to \mathbb{R}$ by setting

$$\nu(\sum_{n\in\mathbb{Z}} a_n q^n) = \max\{n \cdot (\kappa c_N) |\, a_n \neq 0\})$$

and $\nu(x) = 0$ for non-zero $x \in H_*(M,\mathbb{K}) \otimes 1$.

Examples

Let $\dim M = 2n$, α toroidally monotone class, and $\tilde{\phi} \in \widetilde{\mathrm{Ham}}(M)$.

Fixing a homogeneous class in quantum homology $e \in QH_r(M)$, we have a map

$$(e*) : HF_k^t(\tilde{\phi})_\alpha \to HF_{k+r-2n}^{t+\nu(e)}(\tilde{\phi})_\alpha,$$

via intersecting Floer trajectories with a cycle representing e.

$(e*)$ is an operator on Floer persistence module.

Alternatively, we have a map

$$* : HF_*^t(\mathbf{1})_{pt} \otimes HF_*^s(\tilde{\phi})_\alpha \to HF_*^{t+s}(\tilde{\phi})_\alpha,$$

defined by counting pairs of pants. Fixing $e \in HF_r(\mathbf{1})_{pt}$ we obtain the above operator (note that $HF_r(\mathbf{1})_{pt} \cong QH_r(M)$).

Operator interleaving distance

We may define a δ-operator-interleaving by asking for interleaving maps to be morphisms of persistence modules with operators (i.e. to commute with the operators).

$$d_{op-inter}((V,A),(W,B)) =$$

$$\inf\{\delta \geq 0 | \text{ there exists a } \delta\text{-operator-interleaving}\}.$$

Proposition (Polterovich, Shelukhin, S. 2017.)

$$d_{op-inter}((V,A),(W,B)) \geq d_{inter}(\operatorname{im} A, \operatorname{im} B)$$

$$d_{op-inter}((V,A),(W,B)) \geq d_{inter}(\ker A, \ker B)$$

Proof of the proposition follows directly from the definitions.

From now on we work over the field $\mathbb{Q}(\zeta)$, where $\zeta^p = 1$.

$\mathbb{Z}_p$-action

Let (V, A) and (W, B) be $\mathbb{Z}_p$ persistence modules. If a morphism commutes with A and B it also commutes with $A - \zeta \cdot \mathbf{1}_V$ and $B - \zeta \cdot \mathbf{1}_W$. The proposition thus gives us

$$d_{op-inter}((V, A), (W, B)) \geq d_{inter}(\ker(A - \zeta \cdot \mathbf{1}_V), \ker(B - \zeta \cdot \mathbf{1}_W)).$$

Upshot: We may concentrate on interleaving distance between ζ-eigenspaces of $\mathbb{Z}_p$-actions. This allows us to use The isometry theorem and analyse barcodes.

Floer theoretic recipe

- Take $\tilde{\phi} \in \widetilde{\mathrm{Ham}}(M)$, $\phi = \tilde{\phi}_1$.
- Create a Floer persistence module $HF(\tilde{\phi}^p)_\alpha$.
- Introduce $\mathbb{Z}_p$-action by the push forward map $P(\phi)$:

$$P(\phi) : HF^t_*(\tilde{\phi}^p)_\alpha \to HF^t_*(\tilde{\phi}^p)_\alpha, \quad (P(\phi)(z))(t) = \phi(z(t)).$$

- Use continuation maps as operator-interleavings:

$$HF^t_*(F)_\alpha \xrightarrow{C(F,G)} HF^{t+\mathcal{E}^+(G-F)}_*(G)_\alpha \xrightarrow{C(G,F)} HF^{t+\mathcal{E}(G-F)}_*(F)_\alpha.$$

Cook for 20 minutes to get:

$$\tilde{d}_H(\tilde{\phi}, \tilde{\varphi}) \geq \frac{1}{p} d_{op-inter}(HF(\tilde{\phi}^p)_\alpha, HF(\tilde{\varphi}^p)_\alpha)$$

Highlight: If $\phi = \psi^p$, $HF(\tilde{\phi}^p)_\alpha$ has $\mathbb{Z}_{p^2}$-action given by $P(\psi)$. This action shapes the ζ-eigenspace, i.e. we have that the multiplicity of each bar in the barcode of $\ker(P(\phi) - \zeta \cdot \mathbf{1})$ is divisible by p.

Let $\mathrm{Powers_p}(M) = \{\phi \in \mathrm{Ham}(M) \mid \exists \psi \in \mathrm{Ham}(M),\ \phi = \psi^p\}$.

Question: What is $\sup\limits_{\phi \in \mathrm{Ham}(M)} d_H(\phi, \mathrm{Powers_p}(M))$?

Theorem (Polterovich, Shelukhin 2014.)

If Σ is a surface of genus at last 4, $p \geq 2$ any integer and N a symplectically aspherical manifold it holds

$$\sup_{\phi \in \mathrm{Ham}(\Sigma \times N)} d_H(\phi, \mathrm{Powers_p}(\Sigma \times N)) = +\infty.$$

Distance to powers

Fix homogeneous $e \in QH(M)$ and let

$$E_r := e * (QH_r(N)) \subset QH_{r-2n+\deg e}(N).$$

Multiplication by q induces isomorphism $E_r \cong E_{r+2c_N}$ and we may define $2c_N$ Betti numbers

$$b_r(e) = \dim_{\mathbb{K}} E_r,\ r = 0, \ldots, 2c_N - 1.$$

Theorem (Polterovich, Shelukhin, S. 2017.)

Let Σ be a surface of genus at last 4 and N a monotone symplectic manifold. If there exists $e \in QH(N)$ such that $p \nmid b_r(e)$ for some r it holds

$$\sup_{\phi \in \mathrm{Ham}(\Sigma \times N)} d_H(\phi, \mathrm{Powers_p}(\Sigma \times N)) = +\infty.$$

Remark: J. Zhang tackled the case of $\Sigma \times N$ for any closed N and p large enough.

Back to filtered homology

Let f, g be Morse functions, $V^t(f)$ and $V^t(g)$ associated filtered homologies. We saw that

$$\|f - g\|_\infty \geq d_{inter}(V(f), V(g)) = d_{bottle}(\mathcal{B}(V(f)), \mathcal{B}(V(g))).$$

Motto:

Use $d_{bottle}(\mathcal{B}(V(f)), \mathcal{B}(V(g)))$ to distinguish between f and g.

Also

$$\|f - g\|_\infty \geq d_{op-inter}((V(f), e*), (V(g), e*)).$$

which together with the proposition gives

New motto:

Use $d_{bottle}(\mathcal{B}(e * (V(f))), \mathcal{B}(e * (V(f))))$ to distinguish between f and g.

Back to the filtered homology

Question

Can we do better with $d_{bottle}(\mathcal{B}(e * (V(f))), \mathcal{B}(e * (V(f))))$ than with $d_{bottle}(\mathcal{B}(V(f)), \mathcal{B}(V(g)))$? More precisely can we have

$$d_{bottle}(\mathcal{B}(V(f)), \mathcal{B}(V(g))) = 0,$$

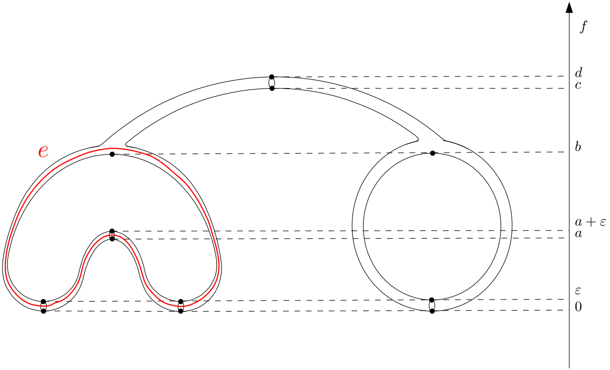
while

$$d_{bottle}(\mathcal{B}(e * (V(f))), \mathcal{B}(e * (V(f)))) > 0,$$

for some class e?

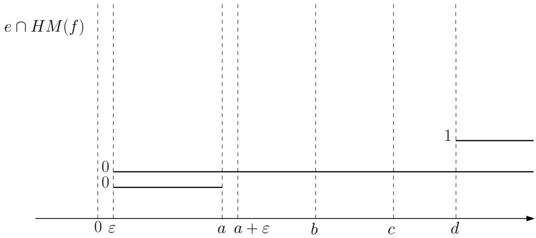
Yes, we can!

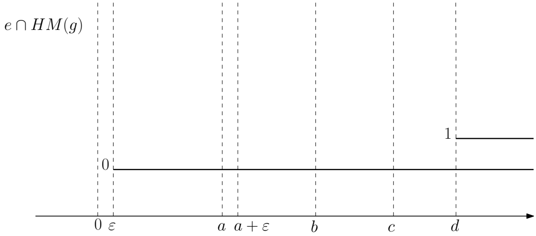
Function f is given as a height function on a deformed surface of genus 2.



Function $g = f \circ \varphi$ is obtained from f by composing with a diffeomorphism of a surface (exchanging left-hand side and right-hand side of the picture).

Barcodes after intersection





Thank you for your attention!