1. Sets and Functions

1.1. Sets.

Definition 1.1. *A set X is a well-defined collection of things, its elements. It is determined by its elements, which can be given as a list or by some description. We write $x \in X$ if x is an element of X. We denote by $|A|$ the number of elements, if A is infinite write $|A| = \infty$.*

In fact, not just every rule works, e.g. the set containing all sets not containing themselves.

Example 1.2.
(1) *All people in this room.* $X = \{\text{ people in this room}\} = \{\text{Name1}, ...\}$
(2) *All real numbers* $\mathbb{R}$.
(3) *The empty set.*
(4) *The set* $\{0, \text{red}, \emptyset\}$.

Definition 1.3. *Subset A is called a subset of B if every element of A is also an element of B. Write $A \subseteq B$.*

Example 1.4.
(1) $\{\text{Name2}, ...\}$ *is a subset of the people in this room.*
(2) $\mathbb{Q} \subseteq \mathbb{R}$.
(3) *Every set is a subset of itself.*
(4) *The empty set is a subset of every set.*
(5) *Often, a subset is given by adding an additional criterion:* $X = \{x \in \mathbb{N} : x \text{ even}\}$.

Definition 1.5. *The power $P(X)$ set of a set X is the set given by all subsets of X.*

Example 1.6. $P(\{0,1\}) = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$

Definition 1.7. *Let A and B be two sets.*
The union $A \cup B$ is the set containing all x such that $x \in A$ or $x \in B$.
The intersection $A \cap B$ is the set containing all x such that $x \in A$ and $x \in B$.
The difference $A \setminus B$ is the set containing all x such that $x \in A$ and $x \notin B$.
The product $A \times B$ is the set containing all pairs (x, y) such that $x \in A$ and $y \in B$.

Example 1.8.
$\{0, \text{red}, \emptyset\} \cup P(\{0,1\}) = \{0, \text{red}, \emptyset, \{0\}, \{1\}, \{0,1\}\}$.
$\{0, \text{red}, \emptyset\} \cap P(\{0,1\}) = \{\emptyset\}$.
$\{0, \text{red}, \emptyset\} \setminus P(\{0,1\}) = \{0, \text{red}\}$.
$\{0,1\} \times \{0,1\} = \{(0,0), (1,0), (0,1), (1,1)\}$.

Definition 1.9. *Suppose I is a set and we have for every $i \in I$ a set A_i, then $\bigcup_{i \in I} A_i$, $\bigcap_{i \in I} A_i$ are the union/intersection of all the sets A_i.*

Example 1.10. $I = \mathbb{N}$, *and* $A_i = [0, n] \subseteq \mathbb{R}$, *then* $\bigcup_{i \in I} A_i = \mathbb{R}$ *and* $\bigcap_{i \in I} A_i = [0, 1]$.

Exercise 1.11.
(1) $A \cup (B \cup C) = (A \cup B) \cup C$.
(2) $A \cap (B \cap C) = (A \cap B) \cap C$.
(3) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
(4) *If A, B finite sets, then* $|A \cup B| = |A| + |B| - |A \cap B|$.
(5) *Calculate the power set of $\{1\}$, $\{1,2\}$, $\{1,2,3\}$ and their number of elements. What is the number of elements of $P(\{1, \ldots, n\})$?*
Hint: One can describe a subset X of $\{1, \ldots, n\}$ by saying for every $1 \leq i \leq n$ if $i \in X$.

2. Functions

Definition 2.1. *Suppose X and Y are sets. A function $f : X \to Y$ is an assignment, which assigns to every $x \in X$ exactly one element $y \in Y$. This y is denoted by $f(x)$. The assignment $f : X \to Y$ can be given explicitly*

as a list or by some rule.
It is possible to assign more than one x to the same $y \in Y$. It is also possible that there is some $y \in Y$ to which no $x \in X$ is assigned.

Example 2.2. (1) $f : \{\text{red}, \text{blue}, \text{green}\} \to \mathbb{R}, \text{red} \mapsto 1.2, \text{blue} \mapsto 107, \text{green} \mapsto -12.$
(2) $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$.

Definition 2.3. *f is called injective if $f(x) = f(x')$ implies that $x = x'$, i.e. if for every $y \in Y$, there is at most one $x \in X$ assigned to y.*
f is called surjective if for every $y \in Y$ there is at least one $x \in X$ assigned to y.
f is called bijective if for every $y \in Y$ there is exactly one $x \in X$ assigned to y.

Example 2.4. (1) *The function $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$ is not injective: For $x = 1$ and $x' = -1$ we have $f(x) = f(x') = 1$, but $x' \neq x$. In other words the "x-values" 1 and -1 are both assigned to the same "y-value" $(-1)^2 = 1^2 = 1$.*

(2) *The function $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$ is not surjective: For $y = -1$, we have for all $x \in \mathbb{R}$ that $f(x) = x^2 \neq -1 = y$. In other words, no "x-value" is assigned to the "y-value" -1.*
(3) *The function*

$$f : [0, \infty) \to [0, \infty), x \mapsto x^2$$

is bijective: Suppose $x, x' \in [0, \infty)$ such that $f(x) = f(x')$. Then $x^2 = (x')^2$, so that $|x| = |x'|$. Since $x, x' \geq 0$, this implies that $x = x'$. Thus, f is injective
Moreover, let $y \in [0, \infty)$. Then $x := \sqrt{y} \in [0, \infty)$ and $f(x) = x^2 = \sqrt{y}^2 = y$. Thus, f is surjective.

Definition 2.5. *Let $f : X \to Y$ be a function. If Y' is a subset of Y then we denote by*

$$f^{-1}(Y') = \{x \in X : f(x) \in Y'\}$$

the preimage of Y'. If $X' \subseteq X$ we denote by

$$f(X') = \{y \in Y : y = f(x') \text{ for some } x \in X'\}$$

the image of X'.

Example 2.6. *Let $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$. Then $f([0, 1]) = [0, 1]$ and $f^{-1}([-2, 1]) = [0, 1]$.*

Definition 2.7. *Let $f : X \to Y$, $g : Y \to Z$. Then $g \circ f : X \to Z, x \mapsto g(f(x))$ is the composition.*

Example 2.8. $f : \mathbb{R} \to \mathbb{R}, x \mapsto x + 1$, $g : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$. *Then* $g \circ f : \mathbb{R} \to \mathbb{R}, x \mapsto (x + 1)^2$.

Exercise 2.9. (1) *Let $f : X \to Y$. Then $f^{-1}(Y) = X$.*
(2) *Let $f : X \to Y$, $Y' \subseteq Y$. Then $f(f^{-1}(Y')) \subseteq Y$.*
(3) *Let $f : X \to Y$, $X' \subseteq X$. Then $X' \subseteq f^{-1}(f(X'))$.*
(4) *Let $f : X \to Y$, $Y', Y'' \subseteq Y$. Show that $f^{-1}(Y' \cap Y'') = f^{-1}(Y') \cap f^{-1}(Y'')$.*
(5) *Let $f : X \to Y$, $X', X'' \subseteq Y$. Show that in general $f(X' \cap X'') \neq f(X') \cap f(X'')$.*
(6) *Let $f : X \to Y$, $g : Y \to Z$, $X' \subseteq X$ and $Y' \subseteq Y$. $g \circ f(X') = g(f(X'))$ and $(f \circ f)^{-1}(Y') = f^{-1}(g^{-1}(Y'))$.*
(7) *Calculate $g \circ f$ for $f : [0, 1] \to [1, 2], x \mapsto x^2 + 1$, $g : [1, 2] \to \mathbb{R}, x \mapsto e^x$.*