BIWEEKLY PROBLEM NO. 35

SEPTEMBER 15 – 30, 2021

Problem. [1] Determine the limit of

$$\frac{\arcsin(\arctan(x)) - \arctan(\arcsin(x))}{\sin(\tan(x)) - \tan(\sin(x))}$$

for $x \to 0$.

[1]This problem goes back to V.I. Arnold and was communicated by Stephan Wagner.

Solution. We determine the limit in the more general case for

$$\frac{f^{-1}(x) - g^{-1}(x)}{g(x) - f(x)}$$

where f, g are two different analytic functions in a neighbourhood around 0 such that $f(0) = g(0) = 0$ and $f'(0) = g'(0) = 1$.

Assume first that $f(x) = x + \sum_{n\geq 2} a_n x^n$ is such a function. Then its inverse is analytic in some neighbourhood of 0 as well, and $f^{-1}(x) = x + \sum_{n\geq 2} \hat{a}_n x^n$. By comparing the expansions for $f^{-1} \circ f$ with id, we obtain

$$x = \left(f^{-1} \circ f\right)(x) = x + (a_2 + \hat{a}_2)x^2 + \dots,$$

hence $\hat{a}_2 = -a_2$ and therefore

$$x = \left(f^{-1} \circ f\right)(x) = x + (a_3 - 2a_2^2 + \hat{a}_3)x^3 + \dots,$$

hence $\hat{a}_3 = -a_3 + p_3(a_2)$ for some univariate polynomial p_3 not depending on f. Recursively, one obtains $\hat{a}_n = -a_n + p_n(a_2, \dots, a_{n-1})$ for $(n-2)$-variate polynomials p_n that are independent of f.

Let now $g(x) = x + \sum_{n\geq 2} b_n x^n$ and $g^{-1}(x) = x + \sum_{n\geq 2} \hat{b}_n x^n$, and let n_0 denote the smallest n such that $a_n \neq b_n$. By the above, this means $\hat{a}_n = \hat{b}_n$ for all $n \leq n_0$, and $\hat{a}_{n_0} - \hat{b}_{n_0} = b_{n_0} - a_{n_0}$. Therefore

$$\frac{f^{-1}(x) - g^{-1}(x)}{g(x) - f(x)} = \frac{\sum_{n\geq 2}(\hat{a}_n - \hat{b}_n)x^n}{\sum_{n\geq 2}(b_n - a_n)x^n} = \frac{(b_{n_0} - a_{n_0})x^{n_0} + \sum_{n\geq n_0}(\hat{a}_n - \hat{b}_n)x^n}{(b_{n_0} - a_{n_0})x^{n_0} + \sum_{n\geq n_0}(b_n - a_n)x^n},$$

which converges to 1 as $x \to 0$.