BIWEEKLY PROBLEM NO. 36

OCTOBER 01 – 15, 2021

Problem. [1] Fix an integer $n \geq 1$ and let $S_n = \{c, c+1, c+2, \dots, c+2^n-1\}$ be a set of 2^n consecutive integers. Show that there exists a partition of S_n into two sets A_n and B_n such that

$$\sum_{a \in A_n} a^i = \sum_{b \in B_n} b^i \tag{1}$$

holds for all $i = 0, \dots, n-1$.

[1]This problem was communicated by Sasha Gasanova.

Solution. For $n \geq 1$, we have $S_1 = \{c, c+1\}$ and the partition $A_1 = \{c\}, B_1 = \{c+1\}$ satisfies equation (1) for $i = 0$ (where we use the convention that $0^0 = 1$). We proceed inductively: Assuming a partition $S_n = A_n \uplus B_n$ fulfills the requirement, let

$$\begin{aligned} A_{n+1} &:= A_n \cup (2^n + B_n) \qquad \text{and} \\ B_{n+1} &:= B_n \cup (2^n + A_n) \end{aligned}$$

where the sum is to be understood as adding 2^n to every element of B_n (A_n, respectively). Observe that this is indeed a partition of S_{n+1}, and moreover, for $i = 0, \ldots, n-1$:

$$\begin{aligned} \sum_{a \in A_{n+1}} a^i - \sum_{b \in B_{n+1}} b^i &= \sum_{a \in A_n} a^i - \sum_{b \in B_n} b^i + \sum_{a \in A_n} (2^n + a)^i - \sum_{b \in B_n} (2^n + b)^i \\ &= \sum_{j=0}^{i} \binom{i}{j} 2^{nj} \left(\sum_{a \in A_n} a^{i-j} - \sum_{b \in B_n} b^{i-j} \right) = 0, \end{aligned}$$

using that the differences of sums over A_n and B_n vanish for all values of i and $i - j$. Finally, it remains to check the sum of n-th powers. A similar computation to the above yields:

$$\begin{aligned} &\sum_{a \in A_{n+1}} a^n - \sum_{b \in B_{n+1}} b^n \\ &\qquad = \sum_{a \in A_n} \left(a^n - \sum_{j=0}^{n} \binom{n}{j} 2^{nj} a^{n-j} \right) - \sum_{b \in B_n} \left(b^n - \sum_{j=0}^{n} \binom{n}{j} 2^{nj} b^{n-j} \right) \\ &\qquad = \sum_{j=1}^{n} \binom{n}{j} 2^{nj} \left(\sum_{b \in B_n} b^{n-j} - \sum_{a \in A_n} a^{n-j} \right) = 0, \end{aligned}$$

thereby finishing the proof.