BIWEEKLY PROBLEM NO. 37

OCTOBER 18 – 31, 2021

Problem. Consider the complete graph on 10 vertices, K_{10}. Is it possible to colour each edge of K_{10} with one of three colours in such a way that all three of the maximal monochromatic subgraphs are (isomorphic to) the Petersen graph?

(The Petersen graph is a graph on 10 vertices with 15 edges, obtained in the following way: Vertices are two-element subsets of $\{1, 2, 3, 4, 5\}$, and two vertices are connected by an edge if and only if the corresponding sets are disjoint.)

Solution. Consider the adjacency matrix (with entry 1 if vertices i and j are neighbours, and 0 otherwise – with vertices labelled $1, \ldots, 10$) of the Peterson graph. Up to conjugating by a permutation matrix, this is

$$P = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

We first determine the spectrum of P: The vector $\mathbf{1} = (1, \ldots, 1)^t$ is an eigenvector with eigenvalue 3 for P. Moreover, observe that $(P^2)_{ij}$ is the number of vertices that are common neighbours to vertices i and j. It can now easily be checked that this means

$$(P^2)_{ij} = \begin{cases} 3 & \text{if } i = j \\ 0 & \text{if } i \text{ and } j \text{ are neighbours} \\ 1 & \text{otherwise} \end{cases}$$

Hence

$$P^2 + P - 2I = J, \tag{1}$$

where I is the identity matrix and J denotes the matrix having 1 in every entry. Since P is symmetric, any eigenvector $v \neq 0$ that is not a scalar multiple of $\mathbf{1}$ is orthogonal to $\mathbf{1}$, hence $Jv = 0$. If $Pv = \lambda v$ it follows from (1) that $(\lambda^2 + \lambda - 2)v = 0$. Thus $\lambda \in \{-2, 1\}$. Additionally, the trace of P vanishes, so the sum of the eigenvalues must be zero. This implies that P has spectrum $(3, 1, 1, 1, 1, 1, -2, -2, -2, -2)$.

Let us now assume that a colouring as required for the problem exists, and let A, B, C be the adjacency matrix of the three monochromatic Petersen graphs. Then

$$A + B + C + I = J. \tag{2}$$

Since both A and B have a 5-dimensional eigenspace for the eigenvector 1, and both these eigenspaces are contained in the (9-dimensional) orthogonal complement to $\operatorname{span}\{\mathbf{1}\}$, there exists a $0 \neq v \in \mathbb{R}^{10}$ such that $Av = v = Bv$. However, equation (2) now implies for this particular v that

$$0 = Jv = Av + Bv + Cv + Iv = 3v + Cv,$$

hence $Cv = -3v$, and therefore -3 is an eigenvalue for the Petersen graph – a contradiction to what was determined earlier!