BIWEEKLY PROBLEM NO. 39

NOVEMBER 16 – 30, 2021

Problem. [1] Define a sequence $x_0, x_1, x_2, \ldots$ as follows: $x_0 = 1$, and for all $n \geq 0$, set $x_{n+1} = \ln(e^{x_n} - x_n)$. Show that the series $x_0 + x_1 + x_2 + ...$ converges, and determine its limit.

[1]This problem was used in 2016's Putnam competition.

Solution. The function $x \mapsto \ln(e^x - x)$ is strictly increasing on $[0, \infty)$, and maps 0 to itself. Thus $x_n > 0$ follows inductively. This in turn implies

$$x_{n+1} = \ln(e^{x_n} - x_n) < \ln(e^{x_n}) = x_n.$$

Hence the sequence $(x_n)_{n \geq 0}$ is bounded from below and monotonously decreasing, and therefore has a limit $x_\infty \geq 0$. This limit needs to satisfy $x_\infty = \ln(e^{x_\infty} - x_\infty)$, which is equivalent to $x_\infty = e^{x_\infty} - e^{x_\infty} = 0$. So $x_n \to 0$ as $n \to \infty$.

By exponentiating the recursion for x_{n+1}, we obtain $x_n = e^{x_n} - e^{x_{n+1}}$. This yields the telescope sum

$$\sum_{i=0}^{N} x_i = \sum_{i=0}^{N} e^{x_i} - e^{x_{i+1}} = e - e^{x_{N+1}}$$

which converges to $e - 1$ as $N \to \infty$.