BIWEEKLY PROBLEM NO. 41

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Problem. Let n be an arbitrary positive integer. Show that there exists a set of n points in the unit square $[0,1]^2$ such that the smallest triangle spanned by three of those points has an area of at least $0.01n^{-2}$.

Solution. Choose points P_1, P_2, P_3 independently uniformly at random from $[0, 1]$. Then, for given $a_2 > a_1 > 0$, we have

$$\mathbf{P}[a_1 \leq \operatorname{dist}(P_1, P_2) \leq a_2] = \pi(a_2^2 - a_1^2) \tag{1}$$

since for this event to happen, P_2 has to be in an annulus around P_1. Given P_1, P_2, for the triangle $P_1P_2P_3$ to have area at most A, the point P_3 must be situated within distance $2A \operatorname{dist}(P_1, P_2)^{-1}$ from the line P_1P_2. This defines a strip of area at most $4\sqrt{2}A \operatorname{dist}(P_1, P_2)^{-1}$ inside $[0, 1]^2$. Moreover, $\operatorname{dist}(P_1, P_2) \leq \sqrt{(}2)$, so by writing x for $\operatorname{dist}(P_1, P_2)$, we obtain from (1)

$$\mathbf{P}[\operatorname{area}(P_1P_2P_3) \leq A] \leq \int_0^{\sqrt{2}} \frac{4\sqrt{2}A}{x} \cdot 2\pi x \mathrm{d}x = 16\pi A \tag{2}$$

Now choose $2n$ points $P_1, \ldots, P_{2n}$ independently and uniformly at random from $[0, 1]^2$. These points span $\binom{2n}{3}$ triangles, and we can label them as $\Delta_1, \ldots, \Delta_{\binom{2n}{3}}$. Define indicator random variables

$$Y_i := \begin{cases} 1 & \text{if } \operatorname{area}(\Delta_i) < \frac{1}{100n^2} \\ 0 & \text{otherwise} \end{cases}$$

Observe that

$$\mathbf{E}[Y_i] = \mathbf{P}[Y_i = 1] \leq \frac{16\pi}{100n^2} \approx 0.503n^{-2}$$

and that $X := \sum_{i=1}^{\binom{2n}{3}} Y_i$ counts the number of triangles that have smaller area than $0.01n^{-2}$. Thus

$$\mathbf{E}[X] = \sum_{i=1}^{\binom{2n}{3}} \mathbf{E}[Y_i] \leq \binom{2n}{3} \frac{16\pi}{100n^2} \leq \frac{(2n)^3}{6} \cdot \frac{16\pi}{100n^2} = \frac{64\pi}{300} n < n.$$

In particular, there exists a configuration of $2n$ points in the unit square with at most n triangles of area $< 0.01n{-}2$. Delete one vertex from each of these small triangles, leaving behind a set of at least n vertices (where additional vertices can be discarded to obtain exactly n vertices) for which the claim holds.