BIWEEKLY PROBLEM NO. 42

FEB 03 – FEB 17, 2022

Problem. Imagine a strip of n stamps, labelled $1, 2, \ldots, n$ in this order, connected by (completely elastic) perforations. Folding together this strip along the perforations and reading off the labels from top to bottom in the resulting, 1-stamp-wide pile yields a permutation of $1, 2, \ldots, n$, and we call a permutation *foldable* if it can be obtained in such a way. Show that a permutation is foldable if and only if there are no two distinct labels s, t of the same parity such that they occur in the order $\ldots s \ldots t \ldots s+1 \ldots t+1 \ldots$, or any cyclic rearrangement of this pattern.

Solution. We deform the problem a bit: Represent the stamps as labelled points along the x-axis of a coordinate plane, and the perforations as circular arcs connecting points i and $i+1$ and running entirely in either of the two half-planes $\{y > 0\}$ and $\{y < 0\}$, depending on whether the perforation between stamp i and $i+1$ is on the left or on the right side of the pile. This geometric representation represents a folding if and only if the arcs don't intersect, and if arcs between $2i$ and $2i+1$ are in the upper, and arcs between $2i-1$ and $2i$ are in the lower half-plane. From this, the necessity of the condition follows immediately.

To see that it is also sufficient, use induction over n. The base case $n = 1$ is trivially true. Assume that a permutation of $n+1$ elements, satisfying the condition, is given. In particular, the condition also holds when the stamp labelled $n+1$ is discarded, and by the induction hypothesis, the first n stamps can be folded accordingly. Now reintroduce the stamp $n+1$, and consider again the geometric picture described above. Assume w.l.o.g. that n is even, then this point $n+1$ can be put inbetween any two consecutive points on the x-axis (say, at $(x_{n+1}, 0)$), unless there is an arch intersecting $\{(x_{n+1}, y) : y > 0\}$ that does not also intersect $\{(x_n, y) : y > 0\}$ where $(x_n, 0)$ are the coordinates of x_n. However, such an arc would connect an even-labelled point s with $s+1$, and its endpoints would be placed in a cyclic rearrangement of the forbidden pattern.