BIWEEKLY PROBLEM NO. 43

FEB 17 – FEB 28, 2022

Problem. Let p be a prime number with decimal digits $p = a_n \dots a_1 a_0$. Show that the polynomial $f = \sum_{i=0}^{n} a_i x^i$ is irreducible in $\mathbb{Z}[x]$.

Solution. By assumption we have $a_n \geq 1$ and $0 \leq a_i \leq 9$ for all $i = 0, \ldots, n$. Suppose $|z| > 1$ and $\mathrm{Re}(z) > 0$. Using

$$\sum_{i=2}^{n} |z|^{-i} = \frac{1}{|z|^n} \cdot \frac{|z|^{n-1} - 1}{|z| - 1} = \frac{1 - |z|^{-n}}{|z|^2 - |z|} < \frac{1}{|z|^2 - |z|}$$

we obtain

$$\begin{aligned} \left|\frac{f(z)}{z^n}\right| &\geq \left|a_n + \frac{a_{n-1}}{z}\right| - \left|\sum_{i=0}^{n-2} a_i z^{i-n}\right| \geq \left|a_n + \frac{a_{n-1}}{z}\right| - 9\sum_{i=2}^{n} |z|^{-i} \\ &> \mathrm{Re}\left(a_n + \frac{a_{n-1}}{z}\right) - \frac{9}{|z|^2 - |z|} \geq 1 - \frac{9}{|z|^2 - |z|} = \frac{|z|^2 - |z| - 9}{|z|^2 - |z|} \end{aligned}$$

and the bound obtained here is positive if and only if $|z| > \frac{1+\sqrt{37}}{2}$. Hence, if α is a zero of f, then $\mathrm{Re}(\alpha) \leq 0$ or $|\alpha| \leq \frac{1+\sqrt{37}}{2}$. In either case, $|10 - \alpha| > 1$.

Assume now that $f = gh$ for two non-constant polynomials $g, h \in \mathbb{Z}[x]$. Then $p = f(10) = g(10)h(10)$, so wlog $g(10) = 1$. Since the roots α_i of g are also roots of f we then obtain from the above that $1 = |g(10)| = \prod |10 - \alpha_i| > 1$, a contradiction.