BIWEEKLY PROBLEM NO. 45

MAR 19 – MAR 31, 2022

Problem. [1] Show that there are no matrices $A, B, C \in \mathrm{SL}_2(\mathbb{Z})$ such that $A^4 + B^4 = C^4$.

[1] This problem was posed as part of SEEMOUS 2009.

Solution. Assume that $A, B, C \in \mathrm{SL}_2(\mathbb{Z})$ with $A^4 + B^4 = C^4$. Denote by a, b, c the traces of A, B, C, respectively. Then, the characteristic polynomial of A (and analogously that of B and C) is given by $\chi_A(\lambda) = \lambda^2 - a\lambda + 1$. By the Cayley-Hamilton theorem, we thus obtain $A^2 = aA - I_2$, and therefore

$$\begin{aligned} A^4 = (aA - I_2)^2 = a^2A^2 - 2aA + I_2 &= a^2(aA - I_2) - 2aA + I_2 \\ &= (a^3 - 2a)A + (1 - a^2)I_2, \quad (1) \end{aligned}$$

together with the analogous identities for B and C. Using (1) for all three matrices yields

$$0 = A^4 + B^4 - C^4 = (a^3 - 2a)A + (b^3 - 2b)B - (c^3 - 2c)C + (1 - a^2 - b^2 + c^2)I_2$$

and after taking the trace on both sides we get

$$0 = a^4 + b^4 - c^4 - 4a^2 - 4b^2 + 4c^2 + 2. \quad (2)$$

Hence we have $a^4 + b^4 - c^4 \equiv 2 \mod 4$, which is only possible if a and b are odd integers, and c is even. Applying this knowledge to (2) again now leads to

$$0 = a^4 + b^4 - c^4 - 4a^2 - 4b^2 + 4c^2 + 2 \equiv 1 + 1 - 0 - 4 - 4 + 0 + 2 \equiv 4 \mod 8$$

which is the desired contradiction.