BIWEEKLY PROBLEM NO. 47

APR 19 – APR 30, 2022

Problem. Without evaluating Riemann's zeta function at the respective points, show that $5\zeta(4) = 2\zeta(2)^2$.

(*Hint:* Consider the expression $f(m,n) := \frac{2}{m^3 n} + \frac{1}{m^2 n^2} + \frac{2}{mn^3}$ for positive integers m, n.)

Solution. Using the hint, we first observe that $5\zeta(4) = \sum_{n\geq 1} f(n,n)$, which, by using telescoping, can also be expressed as

$$\sum_{n,m\geq 1} f(m,n) - f(m+n,n) - f(m,m+n). \tag{1}$$

Considering the summand $f(m,n) - f(m+n,n) - f(m,m+n)$, we find that

$$\begin{aligned}
m^3n^3(m+n)^3\big(f(m,n) - f(m+n,n) - f(m,m+n)\big)& \\
= 2n^2(m+n)^3 + mn(m+n)^3 + 2m^2(m+n)^3& \\
- 2m^3n^2 - m^3n(m+n) - 2m^3(m+n)^2& \\
- 2n^3(m+n)^2 - mn^3(m+n) - 2m^2n^2(m+n)& \\
= (m+n)\big(2n^2(m+n)^2 + mn(m+n)^2 + 2m^2(m+n)^2& \\
- 2m^3(m+n) - 2n^3(m+n) - 2m^2n^2 - m^3n - mn^3\big)& \\
= (m+n)^3\big(2n^2 + mn + 2m^2 - 2(m^2 - mn + n^2) - mn\big)& \\
= 2mn(m+n)^3.&
\end{aligned}$$

Therefore, we obtain

$$f(m,n) - f(m+n,n) - f(m,m+n) = \frac{2}{m^2n^2},$$

and applying this to (1) yields

$$5\zeta(4) = \sum_{m,n\geq 1} \frac{2}{m^2n^2} = 2\left(\sum_{n\geq 1} \frac{1}{n^2}\right)^2 = 2\zeta(2)^2.$$