BIWEEKLY PROBLEM NO. 48

MAY 10 – MAY 24, 2022

Problem. [1] Let n be a positive integer, set $m := (n-1)^2 + 1$ and let π be a permutation of $\{1, 2, \ldots, m\}$, represented as the sequence $\pi(1), \pi(2), \ldots, \pi(m)$. Show that π contains a monotonous subsequence of at least n elements.

[1] This problem was suggested by Baptiste Louf.

Solution. To every index $i = 1, \ldots, m$, we assign two numbers $u(i)$ and $d(i)$ as follows: Define $u(i)$ to be the length of the longest increasing subsequence of $\pi(1), \ldots, \pi(i)$ that contains $\pi(i)$, and $d(i)$ to be the length of the longest decreasing subsequence of $\pi(i), \ldots, \pi(m)$ that contains $\pi(i)$. We claim that $i \mapsto (u(i), d(i))$ is injective.

Indeed, assume there are $i < j$ with $u(i) = u(j) = u$ and $d(i) = d(j) = d$. If $\pi(i) < \pi(j)$ then we obtain a contradiction, as $\pi(j)$ can be appended to any increasing subsequence ending in $\pi(i)$ to produce a longer suibsequence ending in $\pi(j)$, thus $u(i) < u(j)$. Conversely, if $\pi(i) > \pi(j)$ then $\pi(i)$ can be put in front of any decreasing subsequence beginning with $\pi(j)$, and thus $d(i) > d(j)$.

Now assume that the longest monotonous subsequence in π would contain at most $n - 1$ entries. Then all $d(i)$ and $u(i)$ attain values in $\{1, \ldots, n-1\}$, and the map $i \mapsto (u(i), d(i))$ would be an injection from $\{1, \ldots, m\}$ to $\{1, \ldots, n-1\}^{\times 2}$. Since $m > (n-1)^2$, this is absurd.