BIWEEKLY PROBLEM NO. 49

MAY 29 – JUNE 10, 2022

Problem. [1] Consider an ellipse $ax^2 + by^2 = 1$ with rational a, b. Show that if there is a point on this ellipse with both coordinates rational, then there are infinitely many such points.

[1] This problem was taken from T. Andreescu, G. Dospinescu, *Problems from the BOOK*

Solution. Assume (x_0, y_0) is a rational point on $ax^2 + by^2 = 1$. Consider any pair of coprime integers (n, m). We show that the line $(x_0, y_0) + t(n, m)$, where $t \in \mathbb{R}$, intersects the ellipse in a second rational point. Indeed, plugging $(x_0, y_0) + t(n, m)$ into the equation yields

$$ax_0^2 + 2ax_0tn + at^2n^2 + by_0^2 + 2by_0tm + bt^2m^2 = 1.$$

Since (x_0, y_0) lies on the ellipse, we obtain equivalently

$$t(2ax_0n + 2by_0m + t(an^2 + bm^2)) = 0.$$

The solution $t = 0$ only leads to the already known point (x_0, y_0), and the other solution is

$$t = -\frac{2(ax_0n + by_0m)}{an^2 + bm^2} \in \mathbb{Q},$$

and therefore $(x_0, y_0) + t(n, m)$ is a rational point of the ellipse.