**CIM PhD project proposal**

**Elliptic and higher-genus modular graph forms**

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**Introduction:** Recent studies of scattering amplitudes have witnessed a striking confluence of ideas from various areas of physics and mathematics. In particular, the low-energy expansion of string-theory amplitudes became a rewarding laboratory for (elliptic) polylogarithms and modular forms that led to fruitful crosstalk between particle phenomenology, quantum gravity, algebraic geometry and number theory. For loop-level amplitudes of closed strings, the low-energy expansion introduces various flavours of so-called modular graph forms – non-holomorphic modular forms which received considerable attention in the recent mathematics literature. The goal of this CIM PhD project is to study modular graph forms from a variety of perspectives.

**Physics motivation:** String amplitudes serve as valuable testing grounds of string dualities and smoking guns for new structures and symmetries. A particularly tractable setup is to study the S-duality of type-IIB superstrings or heterotic and type-I superstrings at the level of the low-energy effective action which is encoded in the Taylor expansion of scattering amplitudes w.r.t. the inverse string tension $\alpha'$. In the type-IIB case, perturbative information from the $\alpha'$-expansion of gravitational amplitudes led to exact results for duality-invariant curvature interactions of schematic form $\alpha'^3 R^2$, $\alpha'^5 D^4 R^4$ and $\alpha'^6 D^6 R^4$ (with covariant derivatives $D$), see e.g. [1]. Hence, S-duality promotes the low-energy expansion of string perturbation theory to a rewarding window into the non-perturbative regime of string interactions.

$$\int_{M_{0,4}} + \int_{M_{1,4}} + \int_{M_{2,4}} + \int_{M_{3,4}} \ldots$$

As illustrated above, the perturbative expansion of string amplitudes involves integrals over moduli-spaces $M_{n,g}$ of $n$-punctured compact Riemann surfaces of various genera $g$, similar to the loop order of Feynman graphs in field theories. For each external state, one has to integrate over its insertion point on the surface, followed by an integration over its complex-structure moduli (say a single complex modulus $\tau$ for the shape of the torus at genus one). The $\alpha'$-expansion of string loop amplitudes amounts to inserting different numbers of Green functions on the torus or higher-genus surfaces into the moduli-space integrand. Integrating multiple Green functions over the state-insertion points on a fixed surface introduces modular invariant functions of $\tau$, starting with non-holomorphic Eisenstein series in the simplest cases.

These integrals over Green functions dubbed **modular graph forms** [2] go far beyond the literature of classical modular forms and belong to the most exciting exports from theoretical high-energy physics to mathematics of the last ten years. The key objective in this project will be to investigate further generalizations of modular graph forms to so-called elliptic ones and to higher-genus surfaces. By studying the properties of modular graph forms, this project will advance our understanding of string dualities and the interplay of the perturbative & non-perturbative regimes of effective string interactions.

As suggested by the term “modular graph forms”, the $\alpha'$-expansion of string amplitudes admits a graphical organization where each Green function in the integrand is associated with an edge between the state-insertion points. In this sense, modular graph forms can be thought of as discretized Feynman integrals for a conformal scalar on a torus or a higher-genus surface: Modular graph forms are nested sums over discrete lattice momenta which take the role of the continuous loop momenta in field theories on non-compact spacetimes.

**Mathematics motivation:** As a major mathematical value of this project, string amplitudes are studied as generating functions of **single-valued elliptic polylogarithms** – important invariants of mixed elliptic motives with implications all the way to special values of $L$-series of elliptic curves. So far, only the simplest instances of single-valued elliptic polylogarithms are known which calls for the development of a general theory.

The Green functions that drive the $\alpha'$-expansion of string loop amplitudes generalize the logarithm at genus zero to exhibit the periodicity properties of genus-$g$ surfaces. Accordingly, integration over the state-insertion points as prescribed by string perturbation theory introduces various flavours of **polylogarithms**,
certain families of primitives for logarithms. The one-loop order of open-string perturbation theory has for instance been identified as a natural arena for elliptic polylogarithms [3] as defined by Brown and Levin in the recent mathematics literature [4]. In passing from open- to closed-string amplitudes, the (poly-)logarithms entering the $\alpha'$-expansion are replaced by single-valued versions where meromorphic and antimeromorphic quantities are combined such as to cancel local and global monodromies.

At genus zero, Brown’s single-valued polylogarithms constructed in 2004 played a key role in relating closed-string tree-level amplitudes to their open-string counterparts and thereby connecting gauge- and gravitational interactions. As a central mathematical goal of this project, loop amplitudes of closed strings are taken as an inspiration to construct and understand the properties of single-valued versions of elliptic polylogarithms and their higher-genus analogues. More specifically, modular graph forms naturally generalize to the situation where only a subset of the state insertions are integrated over. The resulting functions of one or multiple points on a genus-$g$ surface are dubbed elliptic modular graph forms [5] and expected to span single-valued elliptic polylogarithms in one or several variables. The simplest examples of elliptic modular graph forms are Zagier’s single-valued elliptic polylogarithms [6], and the goal of this project is to explore the infinite families of single-valued elliptic polylogarithms at higher depth and higher genus.

In absence of unintegrated insertion points, modular graph forms may be viewed as single-valued elliptic multiple zeta values which can be described via iterated integrals of holomorphic Eisenstein series. Such iterated-integral representations expose the intricate web of relations among modular graph functions over $\mathbb{Q}$ which are obscured in their more basic definition via nested sums over lattice momenta. Elliptic and higher-genus incarnations of modular graph forms obey even richer collections of $\mathbb{Q}$-relations, so it remains to identify appropriate generalizations of iterated Eisenstein integrals.

**Main goals:** The research objectives of this project can be divided into three mutually reinforcing parts, with more specific targets listed as bulletpoints:

1) understand the systematics of elliptic modular graph forms at genus one in multiple variable
   - represent elliptic modular graph forms in several variables as iterated integrals over modular parameters $\tau$ and insertion points $z$ and develop systematic methods to translate between the two
   - generate a complete list of relations among elliptic modular graph forms in several variables relevant to individual $\alpha'$-orders and make a machine-readable database of results publicly available

2) investigate examples and properties of (elliptic) modular graph forms at genus $g \geq 2$
   - accommodate derivatives of higher-genus Green functions into modular graph forms of higher-genus surfaces and construct generating functions that close under derivatives w.r.t. the moduli
   - explore the web of separating and non-separating degenerations of higher-genus (elliptic) modular graph forms and develop methods to efficiently recover the corresponding lower-genus analogues

3) apply these results to low-energy interactions of gauge- and gravity states in superstring theories
   - extract higher orders in the $\alpha'$-expansion of multiparticle genus-one and genus-two amplitudes of type-IIB gravitons and identify the possible S-duality invariants compatible with old and new data
   - relate gauge-gravity interactions in heterotic-string amplitudes at genus $\leq 2$ to pure gauge interactions in the heterotic theories and the type-I theory; investigate implications for string dualities

**Interdisciplinary aspects:** Modular graph forms and single-valued elliptic polylogarithms promise a rich variety of applications to different areas in high-energy physics. More importantly, as already hinted above, they are also of central interest in number theory & algebraic geometry and for instance found prominent appearance in recent work of Brown and Zagier.

- Brown’s single-valued polylogarithms at genus zero also became a valuable tool in a field-theory context: They vastly simplify the kinematic variables and their appearance in multiloop amplitudes of both dijet scattering in generic gauge theories and the multi-Regge limit of $\mathcal{N} = 4$ Yang–Mills.
- Recent systematic studies of elliptic polylogarithms drastically changed the state-of-the-art methods for the evaluation of Feynman integrals: For many LHC processes, the simplest Feynman integrals beyond the reach of genus-zero polylogarithms crucially improve the theoretical accuracy such that new physics showing up via small deviations in cross sections, distributions and decay rates can be accounted for.
• The mathematical backbone of string perturbation theory is formed by real-analytic functions for closed strings and meromorphic functions for open strings. The interplay of the two cross-fertilizes with powerful concepts in number theory including single-valued integration (over complex variables) [7] and Betti-deRham duality (between integration cycles and differential forms) [8]. By comparing the single-valued elliptic polylogarithms obtained from (elliptic) modular graph forms with the building blocks of the analogous open-string amplitudes, this project can generate essential showcases of single-valued periods. In particular, modular graph forms at genus one exhibit so-called single-valued multiple zeta values among their Fourier coefficients (instead of the algebraic numbers in case of classical modular forms). At genus $g \geq 2$, this project will open up exciting new perspectives on non-holomorphic Siegel modular forms by identifying meromorphic counterparts based on open-string computations.

• The relevance of modular graph forms for algebraic geometry stems from their conjectural match with Brown’s equivariant iterated Eisenstein integrals [9]. The latter correspond to universal mixed elliptic motives in the sense of Hain and Matsumoto, so the results of this project can potentially yield concrete approaches to such motives. The algebraic and differential properties of modular graph forms encode deep arithmetic information on elliptic curves and special values of their $L$-series.

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Advisors: Oliver Schlotterer’s research interests are centered on the interplay of string amplitudes with field theories, string dualities and number theory. His work introduced elliptic multiple zeta values into the physics literature, and he was a key player to relate building blocks of closed- and open-string amplitudes.

Michele Del Zotto’s research is in the context of geometric engineering methods in string theory, as well as string dualities, higher dimensional superconformal theories, and representation theory. In particular, in his works he has studied the details among M-theory and Heterotic string duality via $G_2$ compactifications, and he has found generalizations of mirror symmetry in the context of twisted connected sum $G_2$ manifolds, which would be interesting to explore exploiting the methods in this project.

References


