Espen R. Jakobsen (NTNU)

A convergent discretisation of the porous medium equation with fractional pressure

Abstract: We carefully construct and prove convergence of what is to our knowlegde the first numerical discretisation of the porous medium equation with fractional pressure,

$$\partial_t u - \nabla \cdot (u^{m-1} \nabla (-\Delta)^{-\sigma} u) = 0 \tag{FPE}$$

for $\sigma \in (0, 1)$. The model was introduced by Caffarelli and Vazques in 2011, and is currently one of two main nonlocal extensions of the local porous medium equation. It has finite speed of propagation and comes from a nonlocal Fick's law, but as opposed to the other extension, it does not satisfy the comparison principle. Without comparison, the analysis is difficult. Uniqueness is only known in 1d, where one can exploit that the "cummulative density" $v(x,t) = \int_{-\infty}^{x} u(y,t)\, dy$ satisfies

$$\partial_t v + |\partial_x v|^{m-1} (-\Delta)^s v = 0$$

with $s = 1 - \sigma$. This is a nonlocal quasilinear parabolic equation in nondivergence form that can be analysed through viscosity solution methods.

Our numerical method then loosely speaking consists in discretising this "integrated" equation with a difference quadrature scheme and then compute the solution u of (FPE) via numerical differentiation. Using upwinding in a untraditional way, we obtain a new type of monotone schemes that allows for convergence analysis via the Barles-Perthame-Souganidis half-relaxed limit method. Combining this result with tightness arguments, we then prove convergence of the approximations of the original problem in the Rubinstein-Kantorovich (Wasserstein-1) distance uniformly in time. Our results cover both absolutely continuous and Dirac or point mass initial data, and in the latter case, machinery for discontinuous viscosity solutions are needed in the analysis.

This work is joint with Felix del Teso, UAM, Madrid.

References

[1] Félix del Teso and Espen R. Jakobsen A convergent finite difference-quadrature scheme for the porous medium equation with nonlocal pressure. Preprint: arXiv:2303.05168